Published by Vintage 1997

2 4 6 8 10 9 7 5 3 1

First published in Great Britain by
Jonathan Cape Ltd, 1996

Vintage
Random House, 20 Vauxhall Bridge Road,
London SW1V 2SA

Random House Australia (Pty) Limited
20 Alfred Street, Milsons Point, Sydney
New South Wales 2061, Australia

Random House New Zealand Limited
18 Poland Road, Glenfield,
Auckland 10, New Zealand

Random House South Africa (Pty) Limited
Endulini, 5A Jubilee Road, Parktown 2193,
South Africa

Random House UK Limited Reg. No. 954009

A CIP catalogue record for this book
is available from the British Library

ISBN 0 09 973621 7

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Printed and bound in Great Britain by
Cox & Wyman, Reading, Berkshire

David L. Goodstein
and Judith R. Goodstein

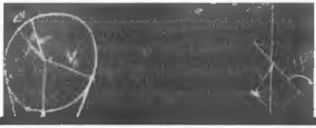
FEYNMAN'S LOST LECTURE

The Motion of Planets Around the Sun

VINTAGE

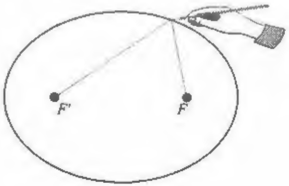
3

Feynman's Proof of the Law of Ellipses

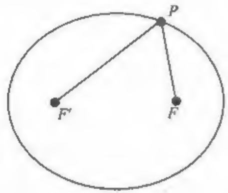


"Simple things have simple demonstrations," Feynman wrote in his lecture notes. Then he crossed out the second "simple" and replaced it with "elementary." The simple thing he had in mind was Kepler's first law, the law of ellipses. The demonstration he was about to present would indeed be elementary, in the sense that it used no mathematics more advanced than high school geometry, but it would be far from simple.

To begin with, Feynman reminds us that an ellipse is a kind of elongated circle that can be made with two tacks, a string, and a pencil, like this:

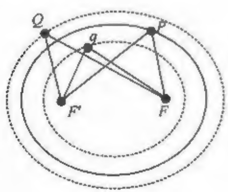


Each tack is at a point called a focus of the ellipse. The string makes a line from one focus to a point on the ellipse and back to the other focus. The total length of the string stays the same as the pencil goes around the curve. Here's a slightly more proper geometric diagram:



Here F' and F are the two foci, and P may be any point on the curve. The distance from F' to P and back to F is the same, no matter where P is on the curve.

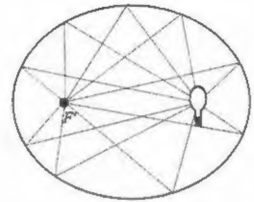
Here is a small point worth remembering: If the string is made a little shorter and the tacks stay where they are, we get another ellipse, inside this one; and if the string is made longer while the tacks stay where they are, we get an ellipse that lies outside this one. It follows that any point in the plane—say, q—such that the distance from F' to q to F is less than the distance from F' to P to F (in other words, any point that can be reached by a shorter string) lies inside our original ellipse. Likewise, any point Q such that $F'Q + QF$ (another way of saying the distance from F' to Q plus the distance from Q to F) is larger than $F'P + PF$ (the length of the original string) lies outside our ellipse. Here's a picture illustrating the idea:



Any point Q outside the ellipse lies on a bigger ellipse, reached by a longer string. Any point q inside the ellipse lies on a smaller ellipse, reached by a shorter string.

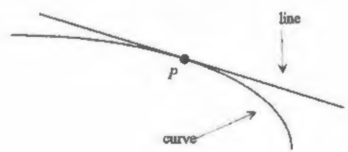
A little later in the discussion, Feynman uses this idea, but he doesn't prove it as we have just done. Instead he asks the students to work it out for themselves.

An ellipse has another special property. If a lightbulb were turned on at F, and if the inner surface of the ellipse reflected light like a mirror, then all the reflected light rays would come back together at F', the other focus, like this:

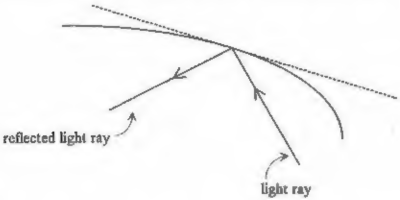


And vice versa: all the light rays starting at one focus will be focused to a point at the other focus. Feynman cites this as the second elementary property of the ellipse, and then he sets out to prove that these two properties are really equivalent. (His strategy here is to lead us to a more arcane property of ellipses—one that will prove indispensable later on.)

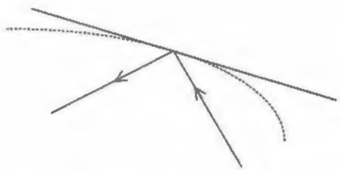
Picture any point P on the ellipse. At that point (as at any point) on the ellipse (or any other curve), there is a single, unique straight line that just touches the curve without penetrating it, like this:



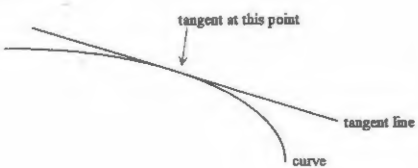
This line is called the tangent to the curve at that point. A light ray, mirror-reflected from the curve at any point, like this,



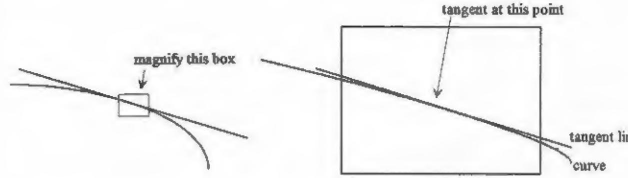
follows the same path it would follow if it were mirror-reflected at that point from the tangent line, like this:



The reason that light reflects from the curve just as it would from the tangent line at the same point is that the tangent indicates the direction of the curve at exactly that point. If one starts with a curve and its tangent at a point,

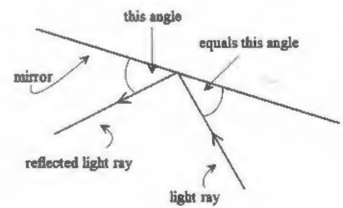


and magnifies the picture greatly about that point, the curve is stretched out to become much more nearly equal to the tangent line:

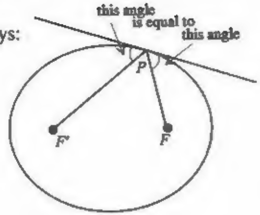


The more closely we look, the less difference there is between the curve and its tangent line at that point. Thus, if light is reflected from a curve at a single point, it reflects just as it would from the tangent line at that point. For the same reason, the tangent line has another property that will be important to us later on: if the curve is actually the path of a moving object, the tangent line shows the direction of the object's motion at each point. When we think of the ellipse as the path followed by a planet in its orbit around the Sun, the tangent to the ellipse at each point will be in the direction of the planet's instantaneous velocity at that point.

The law of reflection from a flat mirror is that the ray strikes the mirror and is reflected from it at the same angle, like this:

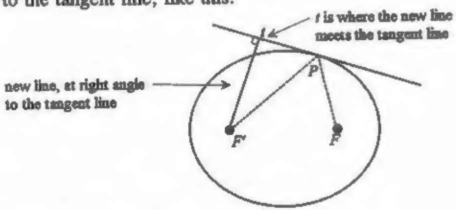


So here is the property for light rays:

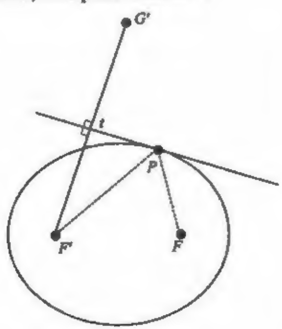


The incident ray from F to P makes the same angle with the tangent line at P as does the reflected ray, which goes to F'. Our job is to prove that this statement is equivalent to saying that the distance $F'P$ plus the distance PF is the same for any point P on the curve.

The proof involves some new construction. A line is drawn from F' perpendicular to the tangent line, like this:

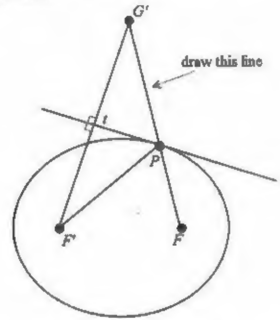


Then it is extended the same distance, to a point called G':

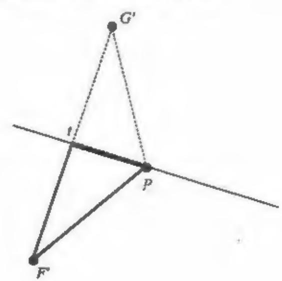


Thus the line $F'G'$ has been constructed in such a way that the line tangent to the ellipse at point P is its perpendicular bisector. Feynman calls G' the image point of F'. What he means is that if the tangent line were indeed a mirror, and if the point F' looked at itself in that mirror, its image would appear to be at G', an equal distance behind the mirror.

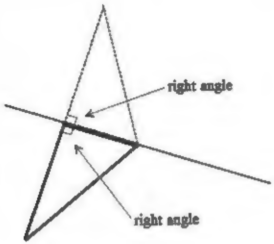
One more piece of construction is called for. Connect the points G' and P with a straight line:



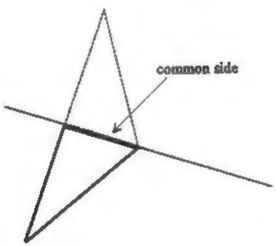
Now take a look at the two triangles that have been formed, one shown with solid lines and the other with broken lines:



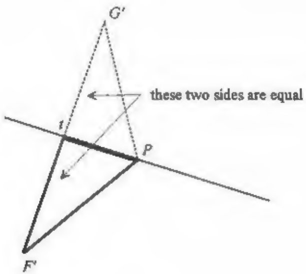
These two triangles are congruent, which means that they are identical in all respects except orientation. Here's the proof. Since we constructed the intersection at *t* to be a crossing of perpendicular lines, each triangle has one right angle:



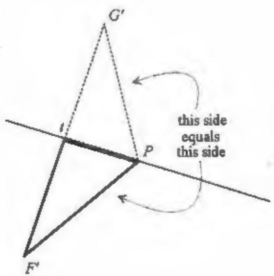
They share one side in common:



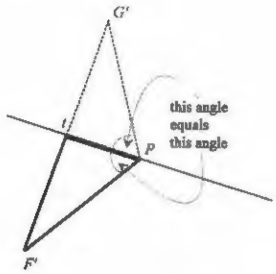
And another side of each was constructed to have equal length (remember, the tangent line bisects $F'G'$:



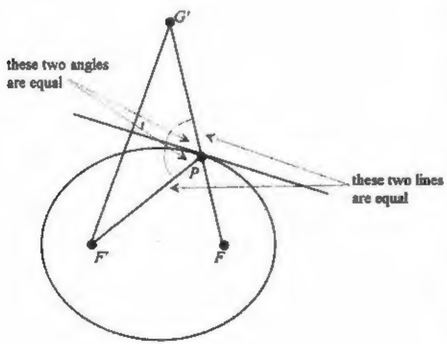
Any two triangles that have one equal angle and two equal sides are congruent; QED, as we used to say in high school. That means *all* the corresponding sides are equal. In particular, note that the side $G'P$ is equal to the side $F'P$:



And the angles $F'Pt$ and $G'Pt$ are equal:

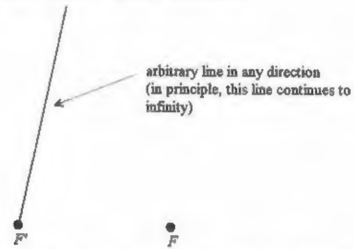


Okay, now back to the full diagram to see what we've learned.

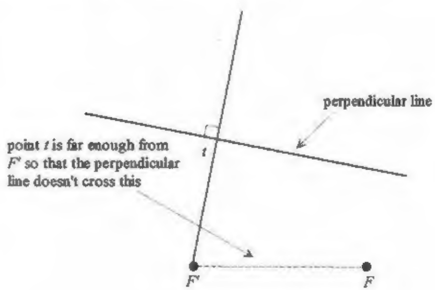


By this time, it's easy to have lost sight of what we're assuming and what we want to prove. To clarify the situation, let's reconstruct the same diagram from scratch. Start with two points, F' and F, that for the moment have no particular significance. They are any two points in a plane:

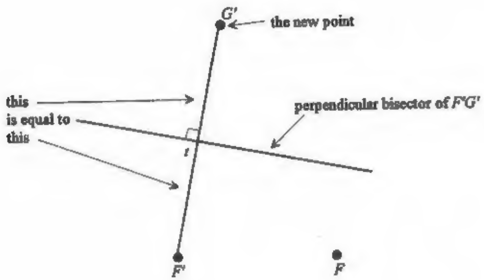
Then draw any straight line from F' in any direction:



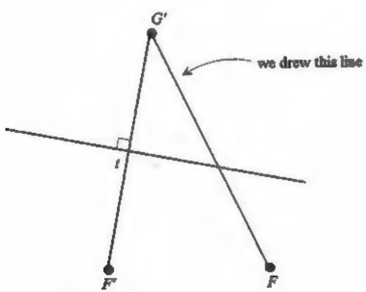
Now pick a point t on the line and draw a perpendicular line through it. The point t must be far enough away from F' so that the perpendicular line doesn't pass between F and F':



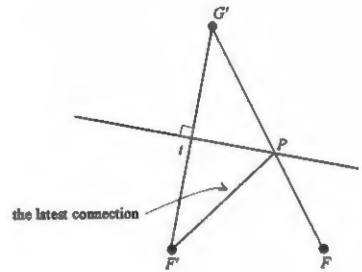
Mark a point G' on the arbitrary line, such that $F't$ is equal to tG'. Then the perpendicular we constructed is the perpendicular bisector of $F'G'$:



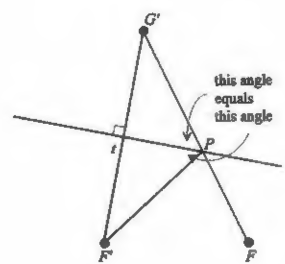
Next draw a line connecting G' and F:



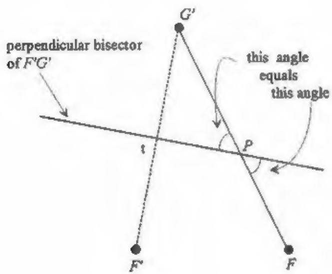
Label the point where this new line crosses the perpendicular bisector P, and draw the line connecting P to F':



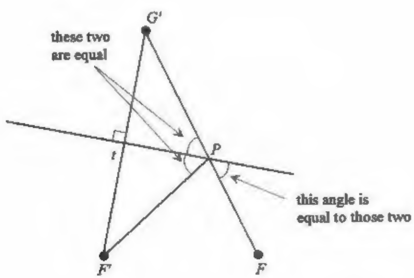
The two triangles are congruent as before, so the angles $F'Pt$ and $G'Pt$ are equal:



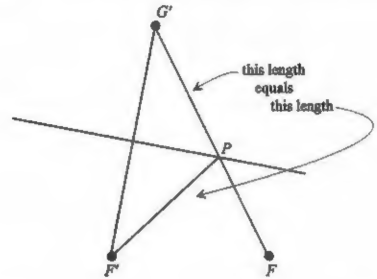
And the angle $G'Pt$ is also equal to the opposite angle where $G'PF$ crosses the perpendicular bisector (when two straight lines cross, the opposite angles are always equal):



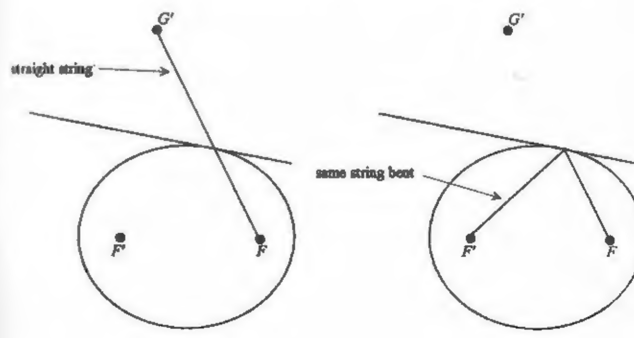
Therefore all these angles are equal:



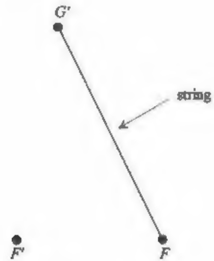
This means the perpendicular bisector line would reflect light from F to F' at the point P (because, at that point, the angles of incidence and reflection are equal). Not only that, the line FPG' has a really spectacular property, which can be seen by going back to the congruent triangles:



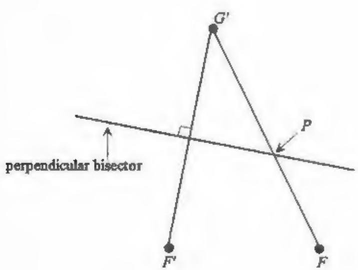
Because of the congruency of the triangles, the length $F'P$ is the same as the length $G'P$. It follows that the distance from F' to P and back to F is the same as the distance in a straight line from F to G'. But that distance is just the length of the string we used to draw our original ellipse. In other words, if we draw an ellipse by the string method, G' is the point we reach by straightening out the string:



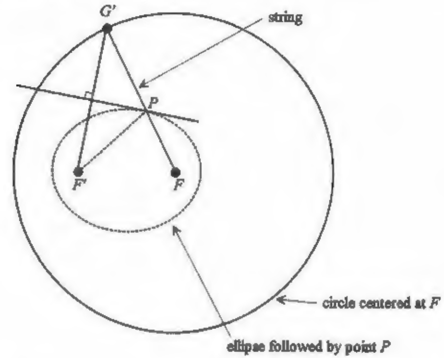
So we've discovered a strange and marvelous new way to construct an ellipse. Here's how it works. Take two points in a plane, F' and F. Then take a string of constant length (larger than the distance $F'F$) and connect one end to the point F. Stretch the string straight in any direction, mark the endpoint, and call it G':



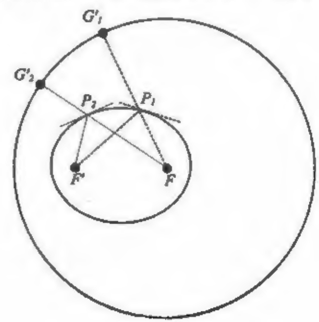
Next, connect F' and G', and draw the perpendicular bisector of $F'G'$. The perpendicular bisector crosses the line FG' at a point P:



Now let the point G' at the end of the string move in a circle of constant radius centered at F:



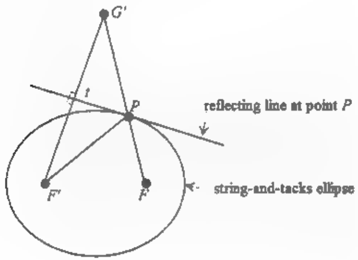
As it does so, the point P, formed by the intersection of FG' and the perpendicular bisector of $F'G'$, traces out exactly the same ellipse that would have been formed using the same string with its ends tacked to F' and F! We know that, because we've proved that when P is constructed in this way, the distance FPG' (which goes from F to the circle) is always equal to the distance FPF' (which constructs the ellipse):



When G' moves from G'_1 to G'_2, P moves from P_1 to P_2, tracing out the ellipse

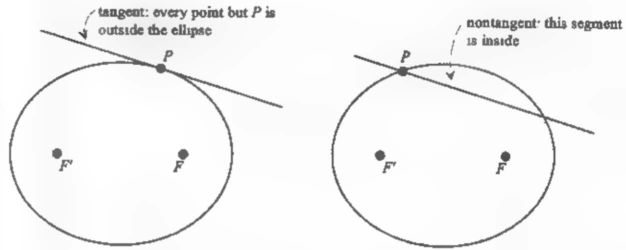
So, within every circle there lurks, for every off-center point, an off-center ellipse. However, while this is very interesting (and will later turn out to be very valuable), it's not the property we set out to prove.

What we did set out to prove is that the string-and-tacks construction of the ellipse is equivalent to its property of reflecting light rays from F to F'. What we have is an ellipse that obeys the string-and-tacks construction (that is, $F'P + PF$ is the same all the way around the ellipse), and the line that reflects light that arrives from F at point P, with equal angles of incidence and reflection, back to F'. That reflecting line happens to be the perpendicular bisector of $F'G'$:

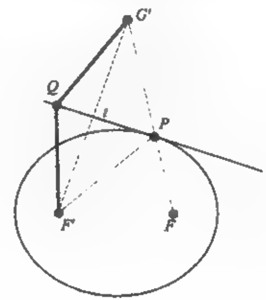


All that's left to prove is that the reflecting line at point P is also tangent to the ellipse at point P. We know that each point on the ellipse has the same mirror-reflection properties as a tangent line at that point. Thus, if the reflecting line at P is also tangent to the ellipse at P, then the ellipse reflects light from F to F' at any point P, and we have proved that the two properties (string-and-tacks and reflecting light from one focus to the other) are equivalent.

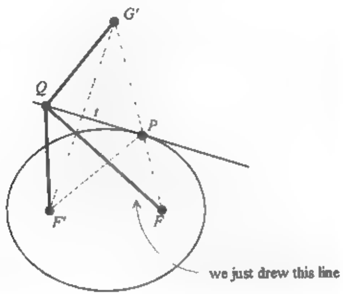
The proof is made by showing that while the point P is (by construction) on both the line and the ellipse, every other point on the line lies outside the ellipse. That is the unique property of the tangent to any curve at a point: it touches the curve without crossing it. If the line crossed the ellipse at P, part of the line would necessarily be inside it:



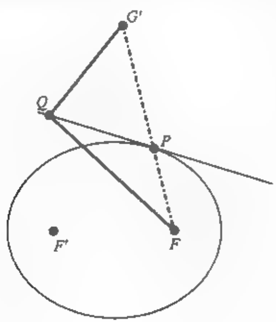
Go back to the construction and pick any point on the line other than P. Label that point Q, and connect it to F' and G':



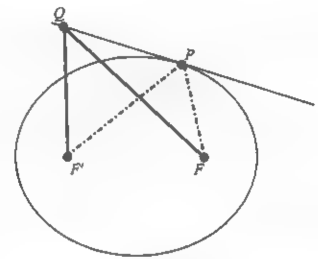
It should be easy to see now that the distances $F'Q$ and $G'Q$ are equal (PQ is the perpendicular bisector of $F'G'$, the triangles $F'tQ$ and $G'tQ$ are congruent, and so on, QED). Now draw the line QF:



The distance from F' to Q to F is equal to the distance from G' to Q to F; we know this because we know that the first steps are equal ($F'Q$ and $G'Q$) and the second steps are the same (QF). Now compare the lengths $FQ + QG'$ (solid lines) and $FP + PG'$ (broken line):



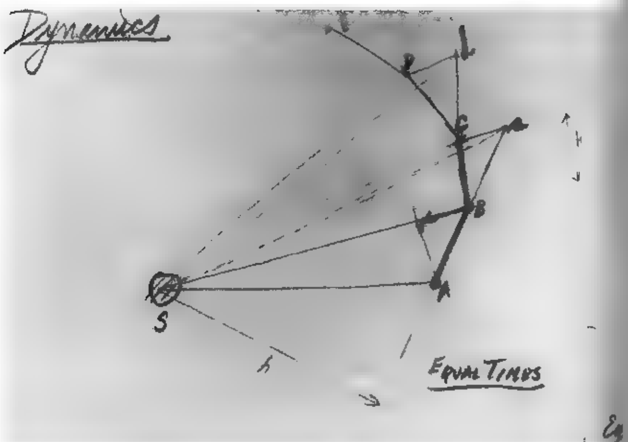
Obviously, FPG' is shorter, since it's a straight line, and a straight line is the shortest distance between two points. But we've just shown that the solid lines $G'QF$ in the drawing above cover the same distance as the solid lines $F'QF$ in the drawing below, and likewise for the broken lines (for the broken lines, we saw that earlier; it's the length of the string):



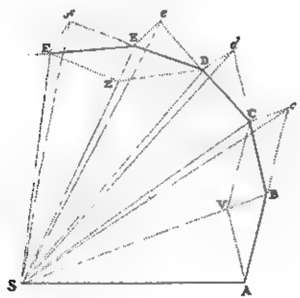
We have proved that the solid lines cover a bigger distance than the broken lines. In other words, if we wanted to reach point Q with a string stretched from tacks at F' and F, the string would have to be longer than the one needed to reach the unique point P. We showed much earlier that this means all such points are outside the ellipse. Thus, the line is tangent to the ellipse at point P. QED.

Speaking of QED, there's something particularly interesting in Feynman's use of this method of proof. We have shown in effect that the shortest path from F' to the tangent line and thence to F is the path that reflects light at point P. This is a special case of Fermat's principle (light always takes the quickest path between two points) and is closely related to Feynman's approach to quantum electrodynamics, which is also known as QED and won him his Nobel Prize. Fermat's principle is a special case of the principle of least action.

In any case, Feynman has now told us all we'll need to know about the ellipse. He turns now to dynamics—that is, forces and the motions that result from them. The diagram that Feynman has sketched in his lecture notes is copied directly out of Newton's *Principia*. That much is obvious from comparing them:

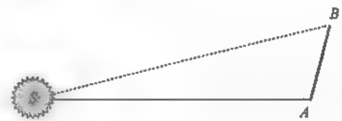


Feynman's Diagram

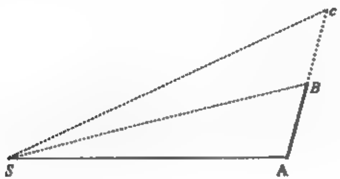


Newton's Diagram

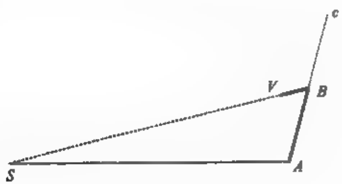
In Newton's diagram, S represents the position of the Sun (the "immovable centre of force"), while A, B, C, D, E, and F are successive positions, at equal intervals of time, of a planet in orbit about the Sun. The motion of the planet is the result of a competition between the planet's tendency to move at constant speed in a straight line if no forces act upon it (the law of inertia) and the motion due to the force that is acting on the planet—that is, the gravitational force directed toward the Sun In reality, these combined effects produce a smoothly curved orbit, but for purposes of seventeenth-century geometrical analysis, Newton represents them by a series of straight-line segments due to inertia, interrupted by sudden changes in direction due to impulsive (essentially instantaneous) applications of the Sun's force. Thus, the first bit of the diagram starts this way:



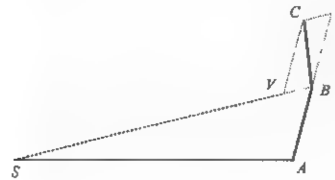
In a certain interval of time, the planet would move from A to B, if no force were acting on it. In the next equal interval of time, if there were no force acting, the planet would continue the straight line for an equal distance, Bc:



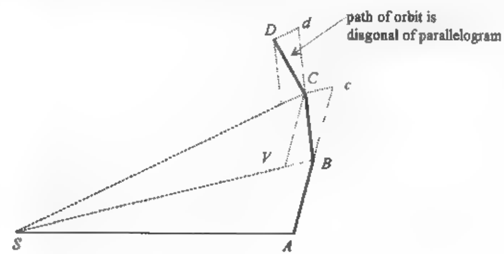
Instead, however, the Sun's force (which really acts continuously) is represented by an impulse applied at point B, which results in a component of motion directed toward the Sun, BV:



The motion the planet would have without the force, Bc, and the motion due to the force, BV, are compounded into a parallelogram; its diagonal is the "actual" motion:

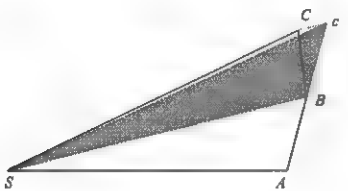


Thus, the planet "actually" follows the path ABC. Notice that Cc is not directed toward the Sun. It is strictly parallel to VB, which *is* directed toward the Sun Incidentally, all of these points lie in a plane: any three points, S, A, B, define a plane. The lines connecting S, A, and B are in the plane. The segment BV lies in the same plane, because it's on the line BS. The segment Bc is in the plane, because it extends the line AB. The line BC is in the plane, because it's the diagonal of the parallelogram formed by BV and Bc. Now the same procedure is repeated at each point, so that the next step looks like this:

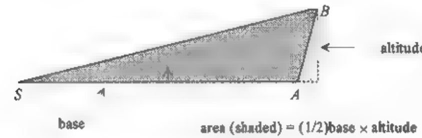


And so on. In the end, Newton applies the same analysis to shorter and shorter equal time intervals, and the resulting path, *ABCD* . . . , becomes arbitrarily close to a smooth orbit, on which both inertia and the Sun's force act continuously The orbit always lies in a single plane.

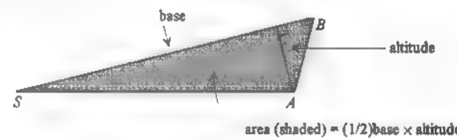
Before shrinking the time interval, Newton (and Feynman) now proves that the planet's orbit sweeps out equal areas in equal times. In other words, the triangle *SAB*, swept out by the planet in the first time interval, has the same area as the triangle *SBC*, swept out in the second equal time interval, and so on. The first step, however, is to show that triangle *SAB* has the same area as *SBc*—a triangle that would have been swept out in the second time interval if there were no force from the Sun. Here's what the three triangles look like:



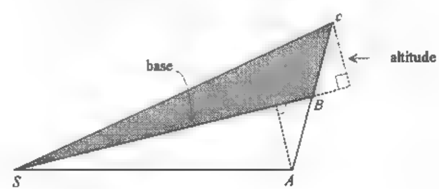
The area of a triangle is equal to one-half its base times its altitude. For example, one way to calculate the area of the triangle *SAB* would be to choose *SA* as the base, in which case the altitude would be the perpendicular distance from the continuation of *SA* to the highest point on the triangle:



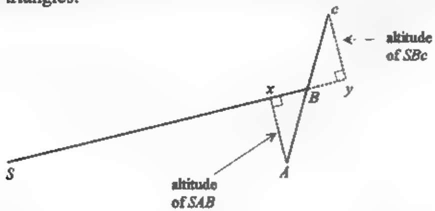
We get the same result if we choose *SB* as the base and construct the altitude like this:



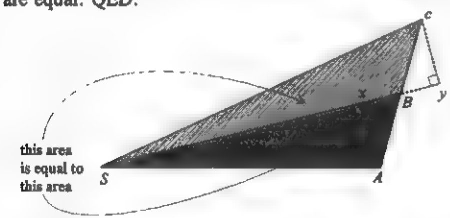
Now we want to compare that area to the area of *SBc*,



where we've chosen *SB* to be the base and constructed the altitude as shown. Look at the diagram formed by the construction of the altitudes of the two triangles:

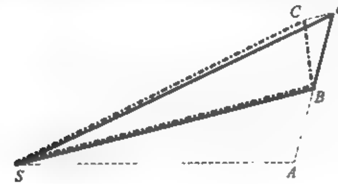


For the moment, the corners where right angles were constructed are labeled *x* and *y*. The triangles *ABx* and *cBy* are congruent, because they have one equal side and two equal angles. The equal sides are *AB* and *Bc* (equal because they are the distances the planet would go in equal time intervals if there were no force from the Sun), and the equal angles are the right angles (*AxB* and *Byc*) and the opposite angles made by the crossing of the two straight lines *xBy* and *ABc*. Since the triangles are congruent, the two altitudes, *Ax* and *cy*, are equal; and since the triangles *SAB* and *SBc* have the same base (*SB*) and equal altitudes, their areas are equal. QED.[1]



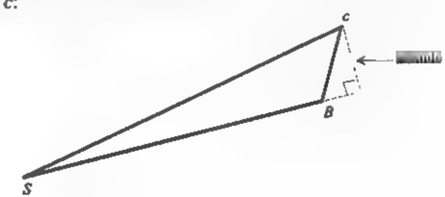
[1]In Feynman's lecture, on page 155, where he does this proof, he chose *AB* and *Bc* as the bases of the two triangles. Then they both have the same altitude, formed by extending the line *ABc* downward, and constructing a perpendicular from it to *S*. This proof and the one in the text work equally well.

Next (following Newton and Feynman), we show that the area of *SBc* (solid lines) is also equal to the area of *SBC* (broken lines):

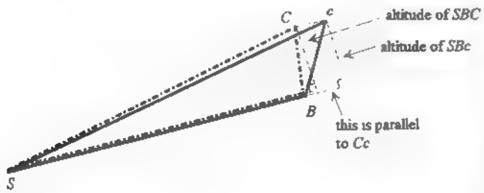


The two triangles have the same base, *SB*. The altitude of *SBC* is the perpendicular distance from the extension of *SB* to *C*:

The altitude of *SBc* is the perpendicular distance from a farther extension of *SB* to *c*:



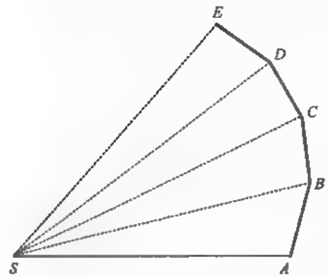
Put the two diagrams back together, and remember that *Cc* is strictly parallel to *SB*:



The two altitudes are the perpendicular distances between the same two parallel lines, and are therefore equal. Thus triangles *SBC* and *SBc* have the same base and equal altitudes. They therefore have the same areas. Once again, QED.

Aside from being very pretty geometry, this last proof is very important for physics. The path *Bc* would have been taken if there were no force at all. Instead, there is a force, directed toward *S*. That force changes the trajectory from path *Bc* to path *BC*, but it cannot change the area swept out during a fixed interval of time. In later years (after Newton but long before Feynman), this area would be understood to be proportional to a quantity called the *angular momentum*. In the language of latter-day physics, we have proved that a force on a planet directed toward *S* cannot change the angular momentum of the planet measured with respect to *S*. Although Newton never used the term "angular momentum," it is clear that he understood the significance of that quantity, and the fact that it could be changed only by a force along some direction not pointing at the center, *S*.

In any case, we have now shown that the area of *SAB* is equal to the area of *SBc* and that the area of *SBc* is equal to the area of *SBC*. It follows that *SAB* and *SBC* have the same areas. Looking back at the original diagram,

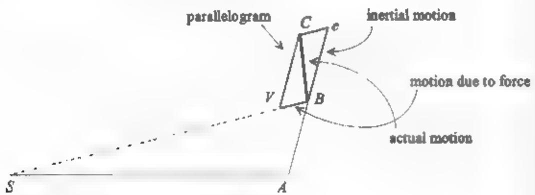


it is obvious that we could apply the same arguments to successive triangles—*SCD*, *SDE*, and so on. These are the triangles swept out by the planet in equal intervals of time. We have thus succeeded in proving Kepler's second law of planetary motion: a planet sweeps out equal areas in equal times.

Now that we can see where we have arrived, it is worthwhile to look back and see how we got here. What exactly did we have to know about dynamics—that is, about forces and the motions they produce—in order to get this far?

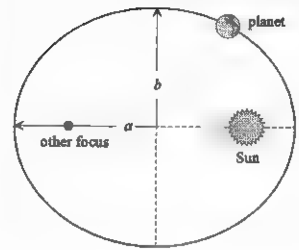
The answer is this: We have used Newton's first law (the law of inertia), Newton's second law (any change of motion is in the direction of the impressed force), and the idea that the force of gravity on the planet is directed toward the Sun. Nothing else. For example, we have not used the idea that the force of gravity is inversely proportional to the square of the distance. So the inverse-square-of-the-distance character of gravity has nothing to do with Kepler's second law. Any other kind of force would have produced the same result, provided only that the force is directed toward the Sun. What we have learned is that if Newton's first and second laws are correct, then Kepler's observation that planets sweep out equal areas in equal times means that the gravitational force on the planet is directed toward the Sun.

You may wonder exactly where we used Newton's first and second laws. We used the first law when we said the planet would move from A to B to c if there were no force on it, and the second when we said that the change in the motion, BV, due to the force from the Sun, is directed toward the Sun. Incidentally, we have also used Newton's first corollary to his laws— that the net motion produced by both tendencies in the time interval is given by the diagonal of the parallelogram of the separate motions that would have occurred:



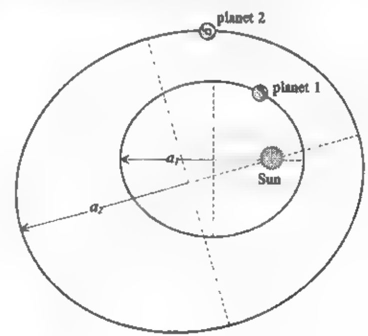
At this point in his lecture, Feynman says, "The demonstration that you have just seen is an exact copy of one in the *Principia Mathematica* by Newton," but he goes on to say that he could not follow Newton's arguments any further, and that he "cooked up" the rest of the demonstration of the law of ellipses himself. Before turning to Feynman's demonstration, however, let us interject another argument that Feynman has disposed of earlier in his lecture: where does the inverse-square-of-the-distance force of gravity come in?

The inverse-square-of-the-distance (from now on we'll just call it the R^{-2}) nature of gravity is deduced from Kepler's third law, which says that the time it takes a planet to make one complete orbit (that is, one year in the life of the planet) is proportional to the 3/2 power of the planet's distance from the Sun. Actually, since the orbits of the planets are ellipses with the Sun at one focus, a given planet is not always the same distance from the Sun:



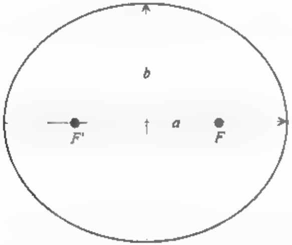
The distance from the center of the ellipse (*not* from the Sun, which is off-center) to the farthest point on the ellipse is called the semimajor axis, labeled a (the shorter axis, labeled b, is called the semiminor axis). The semimajor axis is called that because it is one-half the longest axis of the ellipse. Kepler's third law says that the time it takes a planet to execute one complete orbit is proportional to the 3/2 power of a, the semimajor axis.

Just to be sure the meaning of that statement is clear, imagine a sun with two planets in orbit around it (or a planet with two moons in orbit around it—the same law would be obeyed):

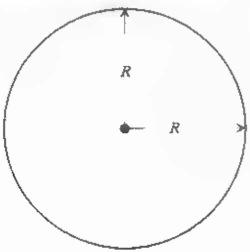


The two arrows show the distances from the centers of the two ellipses to the farthest point of each. Those distances are the semimajor axes, a_1 and a_2. Now suppose that a_2 is twice as big as a_1. Then Kepler's third law says that the time planet 2 takes to make a complete orbit is longer than the orbital period of planet 1 by a factor 2 to the 3/2 power: that is, take 2, cube it to get 8, and take the square root of 8 to get 2.83. The year of planet 2 is 2.83 times longer than the year of planet 1.

The law would still be true, and all the behavior of the planets would be much simpler (but much less interesting), if only Plato had been right and the orbits of the planets were perfect circles. A circle can be thought of as an especially simple ellipse. Starting from an ellipse,



a circle can be constructed by moving both foci, F' and F, to the center:

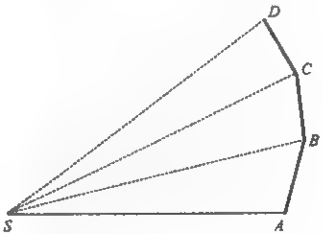


Then the semimajor axis a will be the same length as the semiminor axis b, and we will call both of them the radius, R. Notice that since a circle *is* an ellipse (a special case of an ellipse, to be sure), Kepler's laws allow planetary orbits to be circles but don't require it. In reality, the orbits of the planets in our solar system are all very nearly (but not exactly) circles—although other objects obeying Kepler's laws (such as Halley's comet, for example) have orbits that are very far from circular.

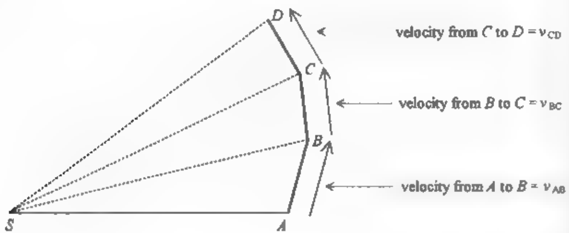
Getting back to our point, we wish to demonstrate that Kepler's third law means that the force of the Sun's gravity diminishes as the square of the distance from the Sun. Following Feynman, we'll simplify the argument by pretending that the planetary orbits really are circles. Symbolically, we'll call the time to complete an orbit T. Then Kepler's third law says $T \sim R^{3/2}$ (read, "T goes as, or is proportional to, $R^{3/2}$"), where R is the distance to the Sun. How is that related to the R^{-2} law?

Like Feynman, we are unable to follow Newton's argument here, and even Feynman's argument is a bit cryptic, so we've formulated our own. This argument is designed not only to make the point about Kepler's third law and Newton's R^{-2} law, but also to introduce some geometrical techniques we'll need for the grand finale.

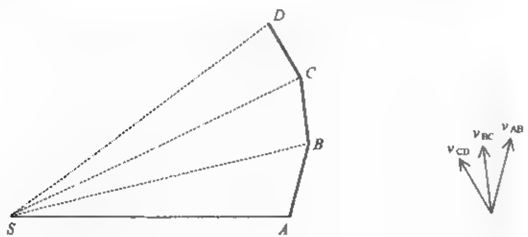
The diagram that we (and Feynman) have copied from Newton shows successive *positions* of a planet in space:



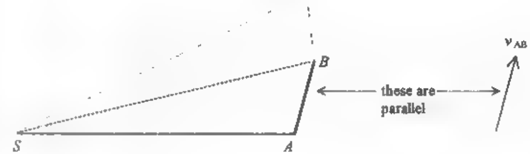
In equal intervals of time, the planet moves from A to B, from B to C, and so on. We can also represent on this diagram the velocity of the planet during each segment (due to inertia, the planet moves from A to B at constant velocity, from B to C at constant velocity, and so on). The velocity can be represented by an arrow pointing in the direction of motion (remember that the word "velocity," as it is used in physics, means not just speed but also direction):



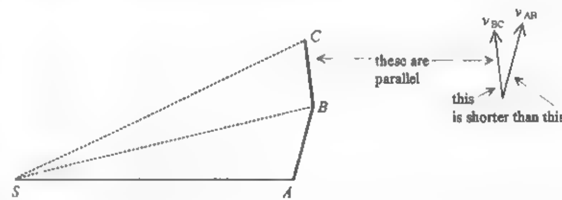
There is no reason for the velocity arrows to be drawn next to the corresponding line segment of the orbit; we can collect them together on the side at a common origin:



The new diagram is a velocity diagram rather than a position diagram. The direction of the arrow shows the direction of the planet's motion, so v_{AB} must be parallel to AB,



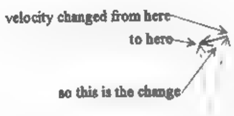
and the length of the arrow is proportional to the speed. In other words, the faster the planet is moving in that segment, the longer the arrow. If the planet moves more slowly on the segment from B to C than it did from A to B, we might get a diagram like this:



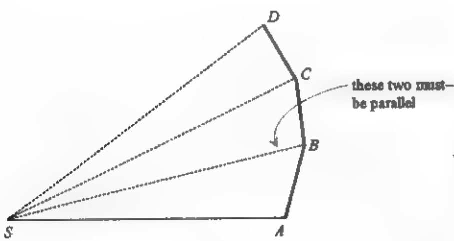
However, the change in velocity, according to Newton's second law, must be in the direction of the Sun, at point B, where the impulsive force causes the velocity to change: If v_{AB} is the velocity before the change,

and v_{BC} is the velocity after the change,

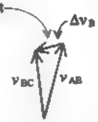
then the change in velocity is also an arrow,



and that arrow must be in the direction of the line from B to S:

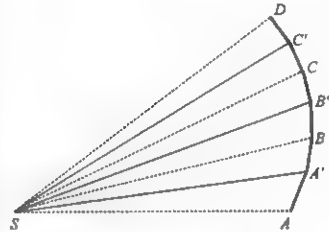


(position diagram)

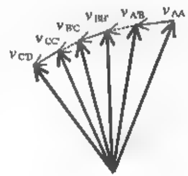


(velocity diagram)

The change in velocity at point B, Δv_B, is thus in the direction of the force from the Sun, and is also proportional to the strength of the force. If the Sun's force were twice as big at point B, Δv_B would be twice as big. That's the meaning of Newton's second law. The change in velocity at each of the (imaginary) points A, B, C, . . . on Newton's diagram also depends on the (equal) time intervals between those points. Newton can (and does) imagine approximating the same orbit by time intervals half as big, to get closer to the actual smooth curve that the orbit makes in space. If all else is the same, and the time intervals are half as big, then each change in velocity will also be half as big but there will be twice as many of them:



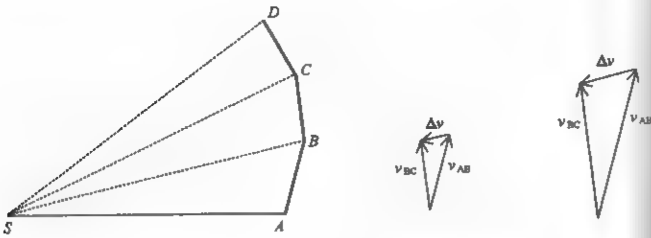
(position diagram)



(velocity diagram)

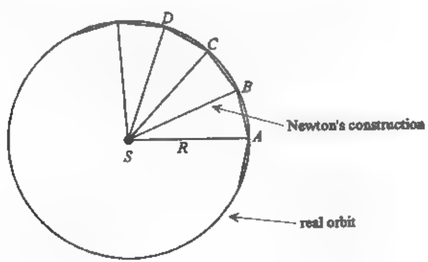
This is the same orbit, produced by the same force as the previous diagram. The force is proportional to the change in velocity at each point (half as big for this diagram) divided by the time interval (also half as big): $F \sim \Delta v/\Delta t$, where F is the force and Δt is the time interval. The force in this diagram is the same as the force in the previous diagram.

There is, as we have seen, an actual correspondence between *direction* on the position diagram and on the velocity diagram. However, the *sizes* of the diagrams bear no relation to one another at all. We could choose to make the entire velocity diagram twice as big (which wouldn't change any of the directions) and it would still be correct:

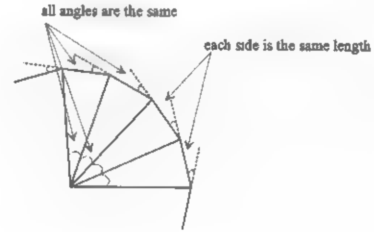


both these velocity diagrams are correct

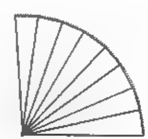
Let's look at the simplest possible specific example. Suppose the orbit were just a circle, of radius R. Then the Newtonian diagram would look like this:



Each of the distances—SA, SB, SC, and so on—would be equal to R, the radius of the circle. Also, each change of velocity, due to the impulsive force at A, B, C, D, and so on, would be the same no matter how the force from the Sun depends on distance, because all these points are at the same distance from the Sun It follows that the speeds along AB, BC, and so on must all be the same, and the lengths of the segments AB, BC, and so on are all the same. That's the only way the orbit can follow the same path, time after time. In other words, the figure drawn by Newton is a *regular polygon*, a figure of equal sides and angles, inscribed in the circle, which is the real orbit.

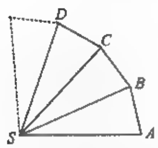


Regular polygons include the equilateral triangle, the square, the pentagon, the hexagon, and so on. The more sides a regular polygon has, the more it resembles a circle Newton imagined using shorter time intervals for his figure, giving a regular polygon with more sides,

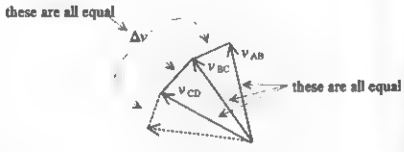


and thus more closely approximating the real circle, ad infinitum, until the real orbit is achieved

In the velocity diagram for a circular orbit, all the velocities are of equal length and at equal angles apart, so that all the changes Δv are the same:

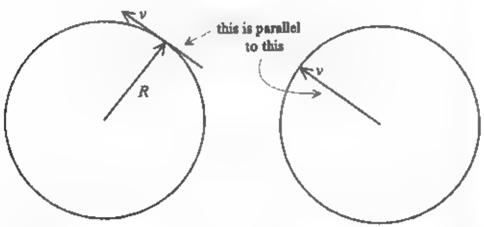


(orbit diagram)



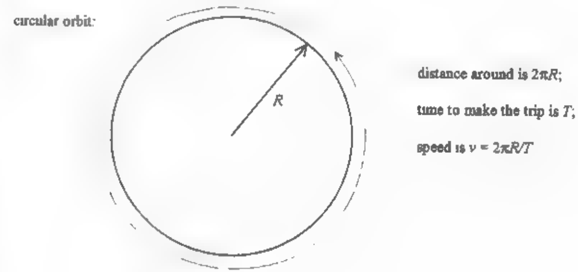
(velocity diagram)

Thus the velocity diagram is also a regular polygon, which also becomes a circle when the orbit becomes a circle (after going through the ad-infinitum):

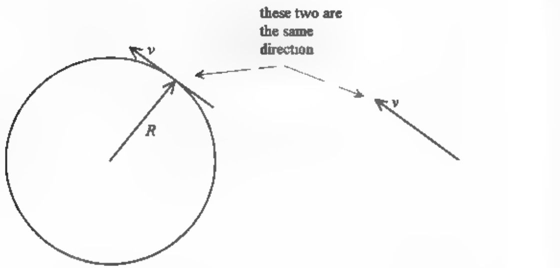


(circular orbit) (circular velocity diagram)

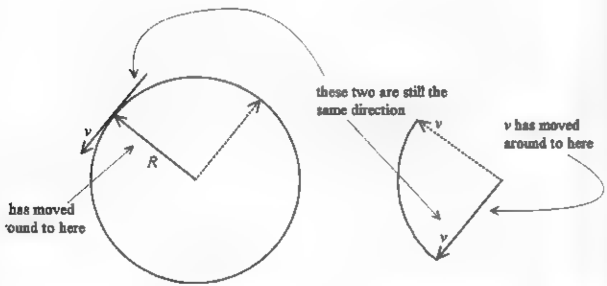
The radius of the circle in the velocity diagram is v, the uniform speed of the planet all the way around its orbit. That speed is given by the distance the planet travels divided by the time it takes. The distance the planet travels is the circumference of the orbit—that is, $2\pi R$—and the time that the planet takes to go around is just what we have called T, the period of the orbit. Therefore, v is equal to $2\pi R/T$.



Each time the planet makes one complete orbit, the velocity arrow also goes around one whole cycle:

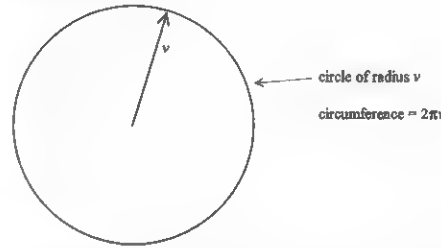


(orbit diagram) (velocity diagram at the same instant)

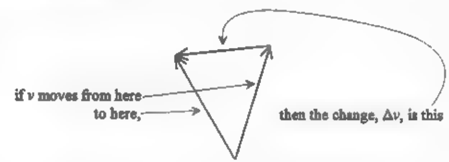


(orbit diagram later) (velocity diagram at the same instant)

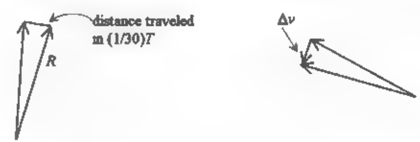
When the velocity arrow makes a complete circle, the tip of the arrow moves a distance $2\pi v$:



Remember that the change in velocity is given by the motion of the tip of the velocity arrow:

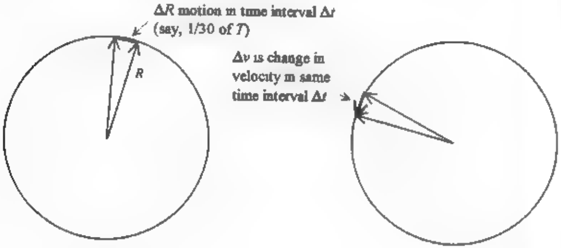


Let's say, now, that the circle has been divided up into 30 parts, each representing the motion in 1/30th of the orbit time T.



(orbit diagram) (velocity diagram)

The force, as we've seen, is proportional to $\Delta v/\Delta t$, where Δv is the change in velocity, equal to 1/30th of the perimeter of the velocity circle, and Δt is the time interval, 1/30th of T. Obviously, 1/30th of the perimeter divided by 1/30th of the time is the same as the whole perimeter divided by the whole time. So $\Delta v/\Delta t$ is equal to the perimeter—that is, $2\pi v$—divided by the time T:



(orbit diagram) (velocity diagram)

$$\frac{\Delta R}{\Delta t} = \frac{2\pi R}{T} \qquad \frac{\Delta v}{\Delta t} = \frac{2\pi v}{T}$$

So the force, F, is proportional to $2\pi v/T$; and the velocity, v, is equal to $2\pi R/T$. Symbolically:

$$F \sim \frac{2\pi}{T} v = \frac{2\pi}{T}\left(\frac{2\pi R}{T}\right)$$

Multiplying the two fractions gives

$$F \sim (2\pi)^2 \frac{R}{T^2}$$

This statement means, for example, that if there were a planet twice as far from the Sun (at $2R$ rather than R) and if it made its orbit in the same time period, then the force on it from the Sun, being proportional to R, would have to be twice as big. However, that's not the way planets behave. We have seen that if there were a planet at $2R$, its period would be $2.83T$. This is determined by Kepler's third law:

$T \sim R^{3/2}$ (the period of a planet is proportional to the 3/2 power of its distance from the Sun)

The force, F, is proportional to the distance, R, divided by T^2. But T^2 means the square of $R^{3/2}$, and $(R^{3/2})^2 = R^3$. So the force is proportional to the distance, R, divided by the cube of the distance, R^3. But R divided by R^3 is the same as 1 over R^2! The force is proportional to 1 over the square of the distance to the Sun! This is the connection we've been looking for—the R^{-2} force law.

Before plunging ahead, this is a good place to stop for a moment, to see where we've been and where we are going.

Kepler has given us three laws, and Newton has given us three laws. Kepler's laws, however, are of a vastly different character from Newton's. Kepler's laws are generalizations of observations of the heavens. They are what we would today call curve-fitting. Kepler took a few points in space—the observed positions of the planet Mars at known times—and said, "Aha! All these points fall on a curve called an ellipse!" That description trivializes the life's work of one of history's great geniuses, but it is nevertheless a correct approximation. That is the essential nature of all three of Kepler's laws.

Newton's laws are of a radically different kind. They are really assumptions about the innermost nature of physical reality: the relations between matter, forces, and motion. If the behavior deduced from those assumptions is observed in nature, then the assumptions may be correct, and if that is the case then we have seen into nature's heart, or the mind of God, depending on your taste in metaphors. In the crucially important arena of planetary motions, the test of whether the Newtonian assumptions are correct is whether they give rise to the Keplerian laws, which summarize with great precision an immense amount of astronomical data.

The connection between Newton's laws and Kepler's laws is more complex than that, however; so far, there is a missing link. In order to determine the planetary motions that his laws would dictate, Newton had to discover the nature of a particular kind of force—the force of gravity. In order to do so, he made use of Kepler's second and third laws Then, having thus deduced the nature of gravity, he was able to demonstrate that the force of gravity, acting under the direction of his laws, would produce Kepler's remaining observation, the law of ellipses. That is the logical sequence of events as presented by Newton in his *Principia*. We now stand at the point in his argument where we have deduced the nature of gravity, making use of Newton's laws and Kepler's second and third laws. Let's review how we did that, before the curtain rises on our final act—Kepler's first law, the law of ellipses.

As applied to planetary motions, Newton's first law, the law of inertia, says that if a planet has no force acting on it, it will remain at rest if it begins at rest, or it will move forever in a straight line at constant speed if it begins in motion. Why it does so is a mystery, although Newton sometimes refers to the mechanism as the planet's "inner force." However, the point with regard to Newton's laws is not to ask *why* they are true, but to ask only *whether* they are true.

Newton's second law says that if there is indeed a force F acting on a planet, its effect is to divert the planet from the straight line that the planet would have followed at constant speed under the influence of inertia. In particular, if a force is applied for a given time interval, Δt, it produces a change in velocity—that is, a departure from the inertial path, Δv, proportional to the force and in the same direction as the force. That means that if twice the force ($2F$) is applied, then twice the change in velocity ($2\Delta v$) is produced. It also means that $2\Delta v$ can be obtained by applying the same force for twice the amount of time ($2\Delta t$). Symbolically, we would write $\Delta v \sim F\Delta t$. It further means that if the force is toward the Sun, the change in velocity must be toward the Sun.

Newton's third law says that forces which operate between different parts of a planet produce no net force upon the whole planet, so that, for purposes of analyzing planetary motions, we can ignore the fact that planets are large complicated bodies and treat them as if they were concentrated at a mathematical point at their centers.

The picture Newton then pursues is that the Sun, assumed to be immovable, applies on the planets a force, gravity, that diverts them

from the inertial straight lines they would otherwise follow and into their actual orbits.

One property of those actual orbits, described by Kepler's second law, is that a hypothetical line connecting the Sun to a planet sweeps out equal areas in equal times as the planet moves around in its orbit. Newton shows, and we have now shown, that the meaning of Kepler's observation is that the force of gravity acts in the direction of the line connecting the planet to the Sun.

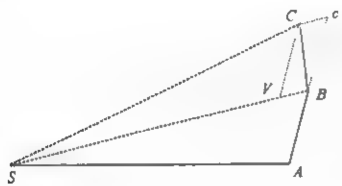
A second property of planetary motion is that the farther away a planet's orbit is from the Sun, the more slowly the planet moves in that orbit. Specifically, the time the planet takes to make one complete circuit increases as the 3/2 power of the distance of its orbit from the Sun. Newton shows, and we have now shown, that to produce this result, the force deflecting the planets into their various orbits must weaken as 1 over the square of the distance from the Sun. In other words, if a planet is twice as far from the Sun, the gravitational force attracting it toward the Sun will be four times smaller.

Notice that Kepler's second law (equal areas) deals with the motion of a single planet in different parts of its orbit, while his third law compares the orbits of different planets. It is strange but true that the masses of the planets have no bearing at all on how fast they move in their orbits. A year (one complete orbit) of the planet Earth is shorter than a year of the planet Jupiter only by the ratio of the 3/2 powers of their distances from the Sun, though Jupiter's mass is more than 300 times that of the Earth.

In any case, we now know that the force of the Sun's gravity on a planet is directed toward the Sun, and that its strength decreases as 1 over the square of the distance from the Sun. We have used Kepler's second and third laws to find out that much. The final, triumphant accomplishment will be to show that such a force of gravity, acting as directed by Newton's laws, will produce elliptical orbits for the planets.

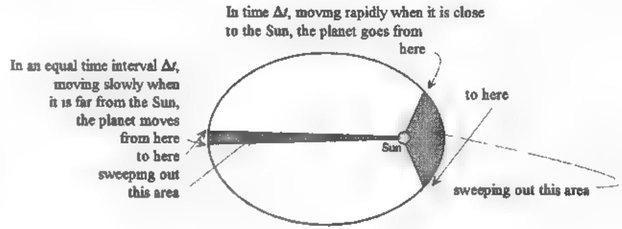
In Feynman's lecture, this is the point at which he finds himself unable to follow Newton's line of argument any further, and so sets out to invent one of his own. His first departure from Newton is much like some brilliant, completely unexpected move by a chess prodigy. Instead of dividing the orbit into imaginary segments that take equal intervals of time, as Newton always does, Feynman divides the orbit into segments that make equal angles at the Sun. We'll need to sketch some diagrams to see what this means.

Recall the diagram in the *Principia* that Feynman copied into his lecture notes:



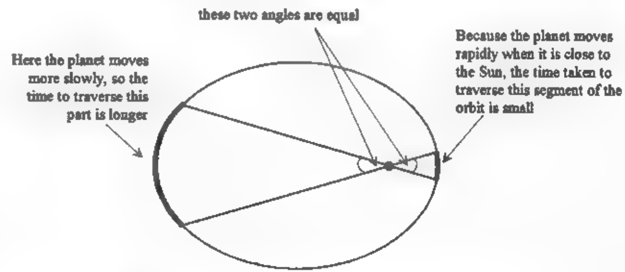
In a certain time interval, a planet would move from A to B if there were no force from the Sun. The time interval might be, for example, 1 second, or 1 minute, or 1 month. In the next equal time interval, it would continue an equal distance from B to c. Instead, the force from the Sun produces an impulse at B that dictates a change in motion, directed toward the Sun, equal to BV. During the second time interval, the planet actually executes a combination of the path Bc, dictated by inertia, and the path BV, dictated by the Sun's gravity: it follows the diagonal of the parallelogram formed by the two motions and arrives at C. We proved earlier that the triangles swept out in equal times, SAB and SBC, have equal areas. Thus Newton approximates the orbit as a series of points equally spaced in time (A, B, C, . . .) at each of which the planet is diverted from its inertial straight line by an instantaneous pull from the Sun. The shorter the time intervals, the more frequent the pull from the Sun and the more nearly the trajectory comes to resemble the real orbit, which is a smooth curve with the Sun's gravity acting continuously to pull the planet away from the inertial straight line it would otherwise have followed. The final, smooth orbit retains the

property we (and Newton and Feynman) have demonstrated for the schematic one: it sweeps out equal areas in equal times, which means that the planet moves faster in its orbit when it is closer to the Sun.



The two areas are equal

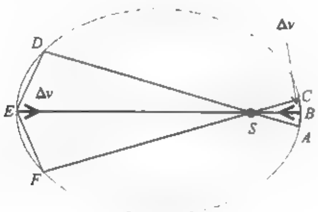
Feynman has used the same argument, taken directly from Newton, to prove this law of equal areas. Now, however, he chooses to divide the orbit into equal angles rather than equal areas:



The two segments of orbit shown above have equal central angles, but they sweep out different areas and therefore take different amounts of time. The law says that the planet sweeps out equal areas in equal times. That means that if it sweeps out half as much area, it takes half as much time, or

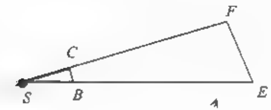
$$\Delta t \sim \text{(area swept out)}$$

Let us for the moment represent these equal-angle segments on a Newton-type diagram, on which the planet undergoes inertial straight-line motions punctuated by velocity changes due to the force of gravity. For simplicity, we draw the velocity changes, Δv, directly on the orbit diagram:

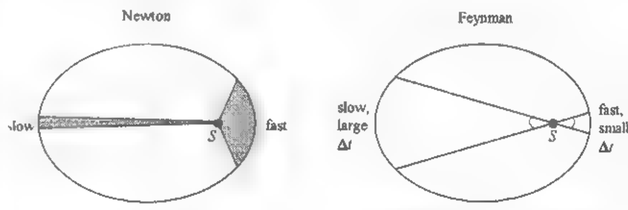


On the side of the orbit closer to the Sun, the planet glides from *A* to *B*, gets diverted by Δv due to the Sun, and continues from *B* to *C*. On the other end of the orbit, the planet goes from *D* to *E*, suffers a pull producing a Δv, and continues from *E* to *F*.

We know that the planet moves faster along *BC* than along *EF*. To see how much faster, we have to compare the areas of the triangles *SBC* and *SEF*, because the times are proportional to the areas swept out. Remember that the two triangles have the same central angle at *S*. Reorienting *SEF* and laying it on top of *SBC*, we have:



The area of each triangle is 1/2 (the base) × (the altitude). Also, these are similar triangles. That means that if the base of the larger triangle is twice as big as the base of the smaller one, then the altitude is also twice as big; in that case, the area of the big triangle would exceed the area of the small one by $2 \times 2 = 4$. The general rule is that the area is proportional to the square of the distance from the Sun.[2] So, the time it takes to go through any portion of the orbit is proportional to the area swept out, which is proportional to the square of the distance from the Sun. Here's a comparison of Newton's way and Feynman's way of dividing the orbit into segments:

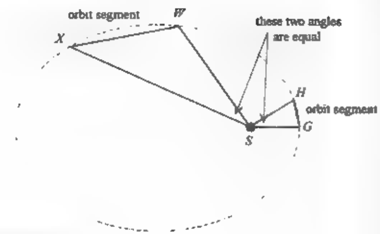


equal times, equal areas

equal angles, $\Delta t \sim$ (square of distance)

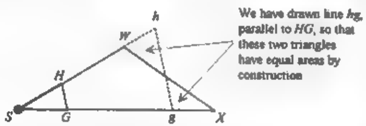
[2] In his lecture, Feynman glosses over this point in a single line. It is not so simple, however, and we haven't really proved it either. Here's a more complete proof. Consider two arbitrary orbit segments that have equal central angles.

(footnote continued)

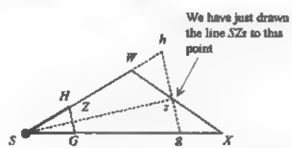


Lay the triangle *SWX* on top of *SGH* like this:

It is always possible to draw a line through *WX*, parallel to *HG*, such that the two little triangles that result will have equal areas.

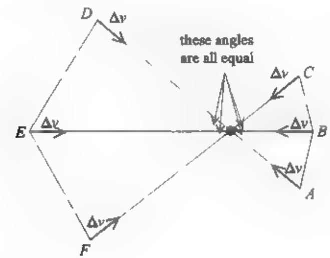


The triangle *Sgh* has the same area as *SWX* (it is bigger by one of the little triangles and an equal amount smaller by the other), and it is similar to *SGH*. Now draw a line from *S* to the point where *WX* crosses *hg*.



We will now call *SZ*, or *Sz*, the distance from the Sun to the orbit. According to the property of similar triangles (base and altitude each increase as the size, so the area is proportional to the square of the size), the similar triangles *SGH* and *Sgh* have areas in proportion to the squares of the lengths *SZ* and *Sz*. But *SWX* has the same area as *Sgh*, so the area of *SWX* is also in proportion

Symbolically, $\Delta t \sim R^2$ in the Feynman drawing, where R is the distance from the planet to the Sun. But we also know that the force from the Sun decreases with distance, according to the inverse-square law—that is, $F \sim 1/R^2$. Let's go back to the kind of diagram that shows the change in velocity, Δv, at each discrete point of the orbit:



At each point around the orbit—*A*, *B*, *C* . . . *D*, *E*, *F* . . . , and all the points in between—there is a Δv toward the Sun. The bigger the force F, the bigger the Δv; also, the longer the time interval Δt, the greater the change in velocity Δv:

$$\Delta v \sim F\,\Delta t$$

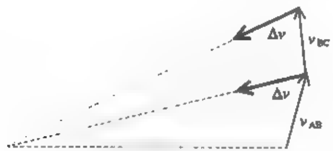
But since $F \sim 1/R^2$ and $\Delta t \sim R^2$,

$$\Delta v \sim (1/R^2) \times R^2 = 1$$

This means that Δv does not depend on R at all! Everywhere in the orbit, no matter how close to the Sun or how far away, the Δv produced in a given *angle* is the same. That happens, as we have now seen,

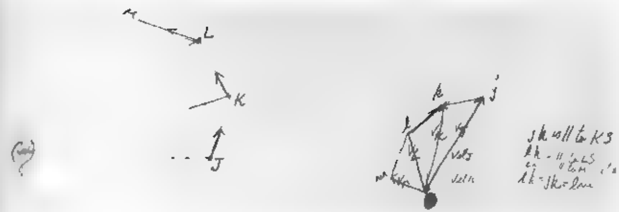
to the square of *Sz*. If we now imagine shrinking the central angle down smaller and smaller ad infinitum, the line *SZz* always stays inside the angle, and because the points *W* and *X* on the elliptical orbit get closer and closer together, the length *Sz* ultimately becomes equal to *SW* or *SX*, which is what we previously called the distance to the Sun. QED because as the planet gets farther away from the Sun, the force acting on it gets weaker (as the square of the distance) but the time the force has to act on the planet gets longer (also as the square of the distance). The result is that all the Δv's, all the way around the orbit, are the same. That, says Feynman in his lecture, is "the central core from which all will be deduced—that equal changes in velocity occur when the orbit is moving through equal angles."

To see exactly what this means, let us look back for a moment at the type of diagram sketched by Newton and copied by Feynman. Rather than representing positions of the planets, we will represent velocities:

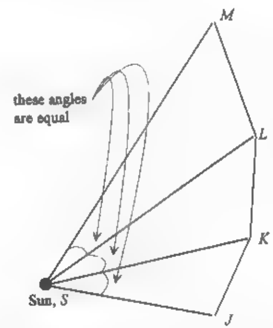


In Newton's way of doing things, the time intervals were all the same, and the Δv's were all pointed toward the Sun, but some Δv's were bigger than others (the biggest Δv's came when the planet was closest to the Sun). In Feynman's scheme, the central angles are all the same, so that the time intervals are different. The Δv's all point toward the Sun (they must, according to Newton's second law) and *they are all now exactly equal in size, all the way around the orbit*. This has consequences that are now to be worked out.

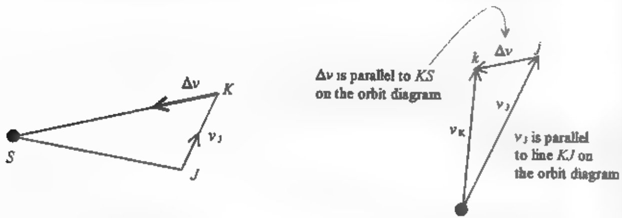
At this point, Feynman has sketched in his lecture notes, with meticulous care, the orbit diagram and the corresponding velocity diagram for equal-angle segments. Here is the result:



The orbit starts from position J, goes to K making some angle at the Sun, suffers a Δv changing its direction, then continues through an equal angle from K to L, and then again from L to M:

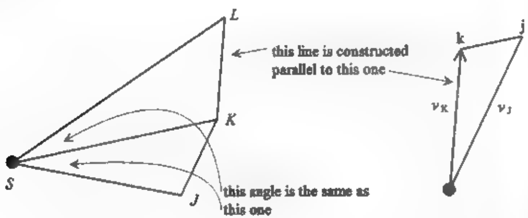


Unlike Newton's version of this diagram, the times of these segments are not necessarily equal. The velocities are in the directions JK, KL, and so on. They are, in general, of different magnitudes on different segments. The changes in velocity suffered at points J, K, L, and M are all directed toward the Sun and all of the same magnitude. In other words, at J there is a Δv in the direction JS; at K, the same Δv occurs in the direction KS; and so on. Using these facts, Feynman constructs the velocity diagram:



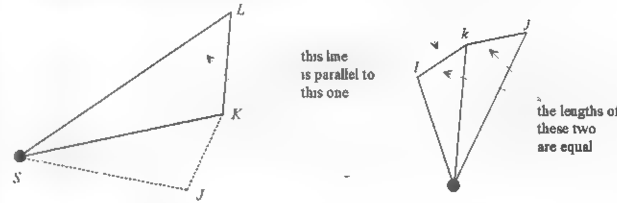
(orbit diagram) (velocity diagram)

On the orbit diagram, the planet moves from J to K with velocity v_J. On the velocity diagram, v_J has the same direction, but not the same length, as JK. At point K, there is a Δv in the direction KS, moving the velocity diagram a distance Δv from point j to point k, where the velocity becomes v_K. This process continues at the next step; the second segment on the orbit diagram is drawn from K, parallel to v_K, to a point L, so that KSL is the same angle as JSK:

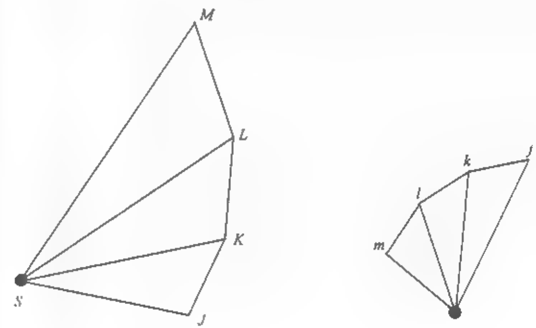


(orbit diagram) (velocity diagram)

We now find the point l on the velocity diagram by adding a Δv equal in magnitude to jk, but parallel to LS:

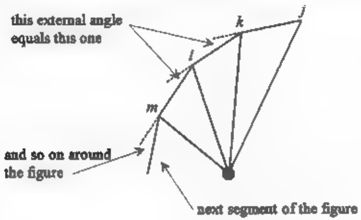


The same procedure can be repeated all the way around the orbit. The next step gives the diagram as Feynman sketched it in his notes:

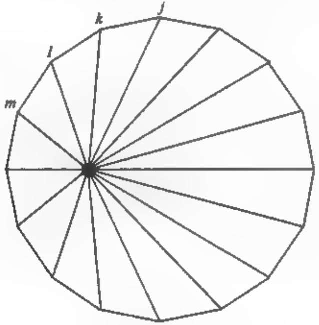


As Feynman wrote in his notes, jk is parallel to KS, lk is parallel to LS, lm is parallel to MS, and $lk = jk = lm$.

Each of the sides of the velocity diagram (jk, kl, lm, . . .) is parallel to one of the lines radiating from the Sun in the orbit diagram. Because the lines from the Sun are constructed to have equal angles, the sides of the figure in the velocity diagram also have equal external angles:



When the velocity diagram is complete, it will be a figure with equal sides and equal (external) angles:

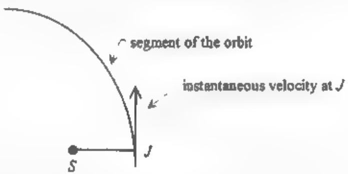


Notice that the velocities themselves, which are the distances from the origin to *j*, *k*, *l*, and so on, are unequal but that the sides (the Δv's) are equal. The resulting figure is a regular polygon! The origin of the

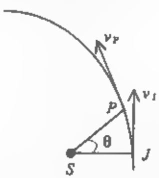
velocities is not at the center, but the external figure itself is a regular polygon.

If we now proceed as usual to divide the orbit diagram into a larger number of segments with equal but smaller angles, the orbit more nearly approaches a smooth curve—and so does the velocity diagram. Because the velocity diagram is a regular polygon, the smooth curve it approaches is a circle! But the origin of the velocities is not necessarily at the center of the circle.

At this point, Feynman sketches in his lecture notes the orbit and velocity diagrams as smooth curves. First the orbit. It starts at point *J*, and Feynman has drawn it in the conventional way, with the line from the Sun extending horizontally; in contrast to the segmented orbit diagram, the velocity at point *J* is a vertical line, perpendicular to the line from the Sun:

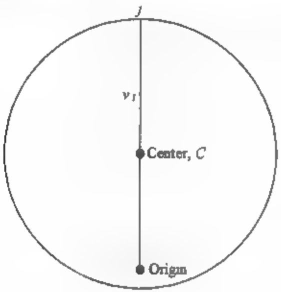


After some time, the planet arrives at point *P*, having made an angle θ at the Sun:

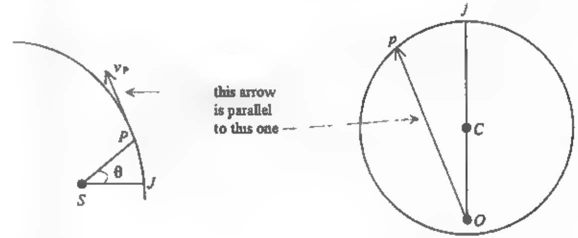


At each point, the instantaneous velocity is tangent to the smooth curve.

Now construct the corresponding velocity diagram. It will be a circle, with the origin off-center. The length of the line we will draw to represent v_J will depend on the planet's speed at point *J* of the orbit. Remember that on a velocity diagram, the longer the line, the faster the speed. Point *J* on Feynman's orbit diagram is also the closest point to the Sun (Feynman has decided this in his head without mentioning it in the lecture), where the orbital speed is greatest. Therefore the line v_J must pass through the center of the circle, because it has to be the longest line on the velocity diagram:

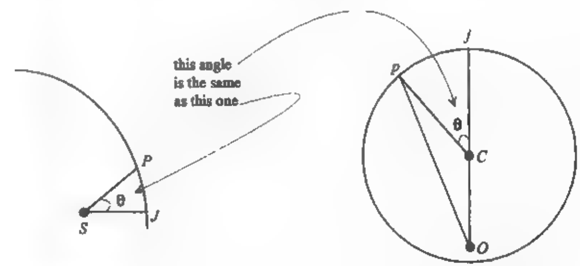


Drawn this way, v_j is vertical (parallel to v_J on the orbit diagram), and it is the longest distance from the origin to any point on the circle. The velocity at point *p* on the velocity diagram, corresponding to *P* on the orbit diagram, is a line from the origin parallel to v_P:



(orbit diagram) (velocity diagram)

It is also true that the angle *jCp* on the velocity diagram is the same angle, θ, as *JSP* on the orbit diagram:



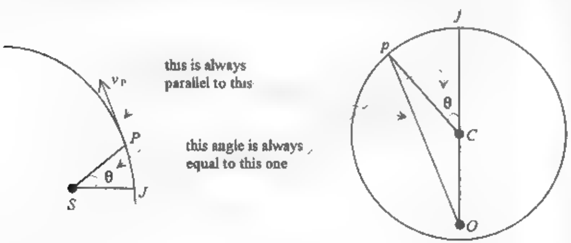
(orbit diagram) (velocity diagram)

The reason for this can be seen if we go back to the complete velocity diagram of orbit segments—the regular polygon—and draw lines out from its center instead of from the origin of the velocity arrows:

(orbit diagram) (velocity diagram)

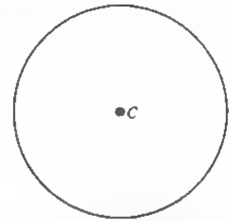
The orbit has been divided up into some number of equal angles, which must total 360°. The polygon necessarily has the same number of equal sides, each occupying the same fraction of 360°. Therefore the angle from *SJ* to any point on the orbit is the same as the angle from *Cj* to the corresponding point on the velocity diagram.

The net result is shown in the pair of diagrams sketched by Feynman:



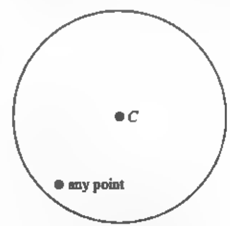
(orbit diagram) (velocity diagram)

Now that all the correspondences between the two diagrams have been established, we could construct the orbit starting from the velocity diagram. It is an easier starting point, because we know that it is just a circle:



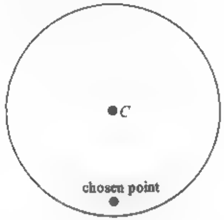
(velocity diagram)

Any orbit permitted by Newton's laws and the force of gravity will have this same velocity diagram. The exact shape of the orbit will depend on where we choose to place the origin of the velocities. Pick a point, any point, inside the circle, but not at *C*, the center (we will see later what happens if the point *is* at *C*, or on the circle, or even outside it):

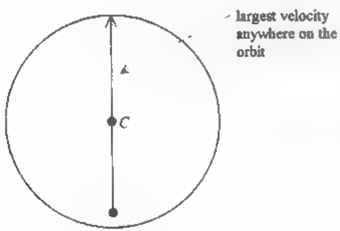


(velocity diagram)

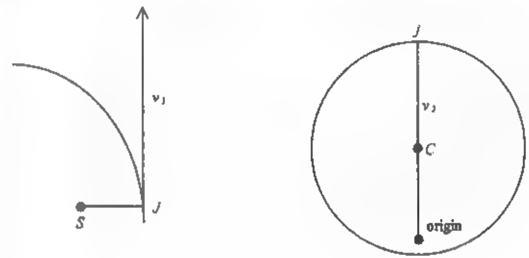
For purposes of familiarity only, turn the whole diagram until the chosen point lies directly below *C*:



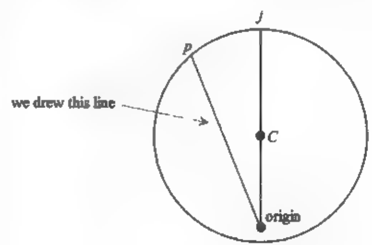
The chosen point is to serve as the origin of velocities: that is, a line from there to any point on the circle's perimeter will have a length proportional to the planet's speed at that point on the orbit, and lie in the same direction as the planet's motion at that point on the orbit. As noted, the line from the origin through the center to the circle's perimeter is the longest line and therefore represents the point on the orbit where the planet is moving fastest.



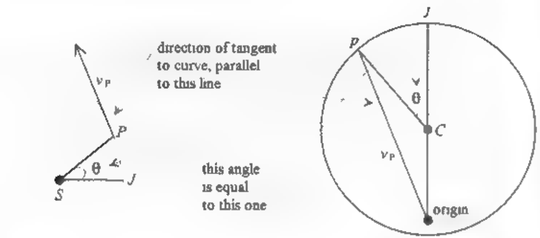
According to the equal-areas law, this will be the point on the orbit closest to the Sun. As Feynman has done, we will draw the orbit so that the line from there to the Sun is horizontal and the velocity is vertical (that's why we rotated the origin of the velocity diagram to be beneath the center):



Now draw a line from the origin to any other point on the circle, *p*:



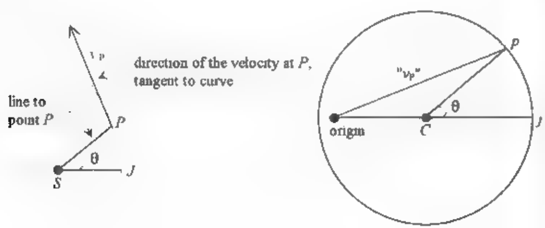
This point corresponds to a point P on the orbit that has the following properties: the line from the origin to p on the velocity diagram is parallel to the tangent at the point P on the orbit diagram, and the angle JCp is the same as the angle JSP·



(orbit diagram) (velocity diagram)

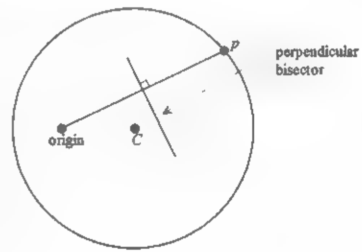
So at each angle θ, we know the direction of the tangent to the orbit we are seeking to construct. How can we construct the curve?

Later in the lecture, Feynman tells us that this was the most difficult step to discover. The trick is to rotate the velocity diagram clockwise by 90°, so that the directions on it are the same as those on the orbit diagram:

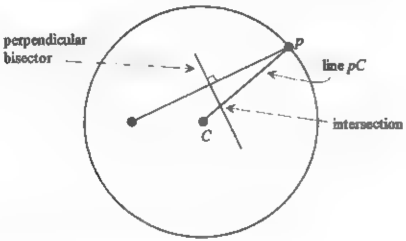


Now the central angle θ is the same on both diagrams, but the line marked "v_P," which was parallel to the velocity at P on the orbit, is now perpendicular to it, since we rotated the whole velocity diagram by 90°. We now know, from the velocity diagram, the direction from the Sun to point P on the orbit, and we know the direction of the tangent to the orbit at that point. It is perpendicular to the line marked "v_P." But we don't yet know exactly where the point is.

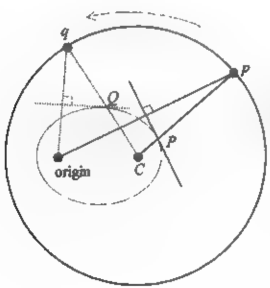
The easiest way to construct the curve having all the required properties is to draw it right on top of the velocity diagram. Then the size of the orbit will be arbitrary, but all the directions, and therefore the shape of the orbit, will be correct. To get the orbit, simply construct the perpendicular bisector of the line from the origin to p:



Because it is perpendicular to the line from the origin to p, we know that it is parallel to v_P, the velocity at point P on the orbit. At some point, the perpendicular bisector crosses the line connecting p to the center, C:

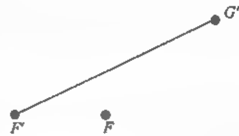


As the point p moves around the circle, the intersection of pC and the perpendicular bisector moves around in a curve of its own:

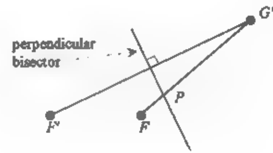


As p moves around the circle to q, the intersection of the construction moves from P to Q and so on, creating the orbit.

We once before made exactly the same construction. Starting from two points in the plane called F' and F (corresponding respectively to origin and C), we drew a line from F' to a point G' (p in the new diagram):

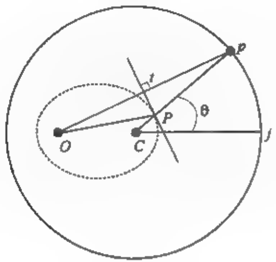


Then we connected $G'F$, and drew the perpendicular bisector of $F'G'$, which crosses FG' at the point P:



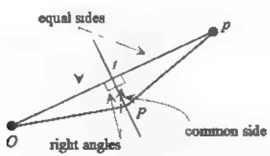
We proved then that as the point G' executes a circle centered at F, the point P executes an ellipse, and at each point P the perpendicular bisector is tangent to the ellipse (see pages 73 to 80).

We have now made exactly the same construction again as on page 79—only the names have been changed. Here's how the new diagram looks:

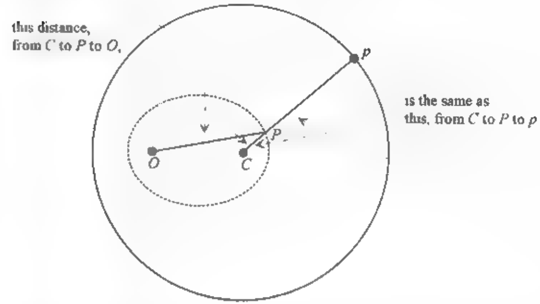


Here, p is a point on a circle centered at C. There is also an eccentric point: the origin of the velocity diagram, which we now call O. The line segment Op has a perpendicular bisector at t, which intersects the line Cp at a point P. We will now prove again that each point P created in this way, as p moves around the circle, lies on an ellipse, and that the line tP is tangent to the ellipse at P. Since tP is parallel to the velocity of the planet when it is at point P on its orbit, we will have constructed the unique curve that has the planet going in the right direction at every point in its orbit.

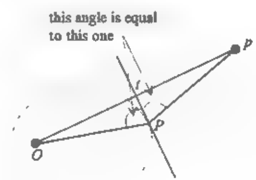
To prove that the curve is an ellipse, we notice that the triangles OtP and ptP are congruent:



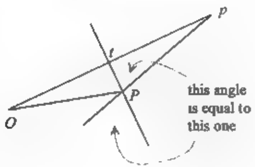
Therefore $OP = pP$. And in the full diagram,



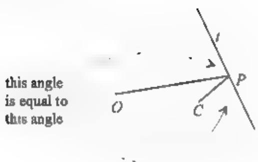
CPp, which is the radius of the circle and is therefore the same all the way around, is equal to $CP + PO$, the length of the string from foci C and O that constructs the ellipse. The dashed curve (the orbit) is therefore an ellipse, QED. To prove that tP is the tangent line at P, go back to the congruent triangles:



Now let the lines Pp and tP cross each other:



Therefore,



The line tP is therefore the line that reflects light from C to O at point P. We long ago proved that the line tP that has that property is the tangent line. For the last time, QED.

The proof is now complete. Feynman is not quite finished yet, but we have accomplished in full what we set out to show. Newton's laws, together with an R^{-2} force of gravity toward the Sun, result in elliptical orbits for the planets Before we leave the subject, let us look back one more time at the logic of the arguments that have enabled us (with the help of Newton and Feynman) to accomplish that heroic feat.

Newton says something like this: From the fact that planets sweep out equal areas in equal times, I used my laws to deduce that the force of the Sun's gravity on a planet points directly toward the Sun. Then, from the fact that the orbital periods of planets are proportional to the 3/2 power of their distances from the Sun, I used my laws to deduce

that the force of gravity diminishes as R^{-2}. Finally, my laws, together with these two facts about gravity, produce elliptical orbits.

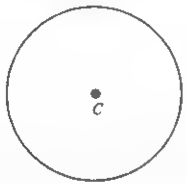
Newton didn't really think about the problem that way. We know from earlier versions of his work (for example, the brief treatise he sent to Halley in 1684) that he experimented with various forms of his axioms about dynamics. Only later did he reduce them to three and start to refer to them as "laws." The act of reducing all of dynamics to three fundamental laws was supremely important, because, as Newton and his followers were to show over the course of the ensuing three centuries, those laws could be used to explain not only the motions of the planets but almost every other phenomenon in the physical world as well. Newton's laws tell us how matter behaves when it is acted on by forces. The only two things we need to know about the physical world that Newton's laws don't tell us are: What is the nature of matter? What is the nature of the forces that act between bits of matter? These two questions are still the central concerns of the science of physics.

This whole powerful reorganization of our understanding of the world begins with the proof of elliptical orbits. In this case, we do not need to know very much about the nature of matter, because gravity affects all matter in exactly the same way. The nature of the force of gravity is very important, however, and that's what Newton uses two of Kepler's laws to deduce.

Finally, we have seen the proof of elliptical orbits not as Newton originally did it but as Richard Feynman worked it out. Feynman divides the orbit into equal angles. In each equal-angle segment, the change in velocity is directed at the Sun, and proportional to the strength of the force and the time over which the force acts. That is Newton's second law. The time is proportional to the area swept out, which (by pure geometry) is proportional to the square of the distance, and the force is inversely proportional to the square of the distance (that's the nature of the force of gravity); so no matter what the shape of the orbit is, and no matter how close to or far from the Sun the planet wanders, the planet undergoes equal changes of velocity in equal angles. It follows immediately that the velocity diagram is a regular polygon (equal sides at equal angles), which becomes a circle for smooth orbits. However, the origin of the velocity diagram *is not at the center of the circle.*

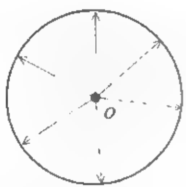
Then, with the help of a geometric construction that has been cunningly set up in advance, it is shown that the orbit has the shape of an ellipse, with the origin of the velocity diagram and the center of the velocity circle acting as foci.

The velocity diagram is a powerful geometric tool. Newton's dynamical laws, together with an R^{-2} force, always produce a circular velocity diagram:



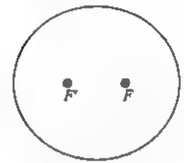
velocity diagram
due to R^2 law

The shape of the orbit depends on where O, the origin of the velocity diagram, is. If O coincides with C, the center of the diagram, then the two foci of the ellipse coincide and the planet has the same speed in all parts of its orbit:

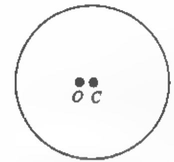


In this case, the orbit is simply a circle.

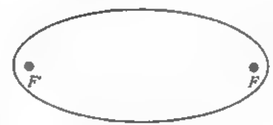
If the point O is anywhere between C and the circumference of the diagram, then the orbit is an ellipse. The closer O is to C, the more nearly circular is the ellipse. The farther O is from C, the more elongated the ellipse:



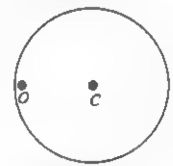
nearly circular orbit



velocity diagram
(turned 90°)



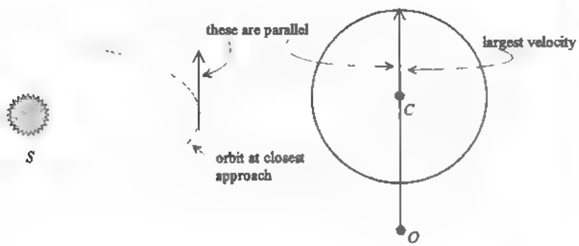
very eccentric orbit



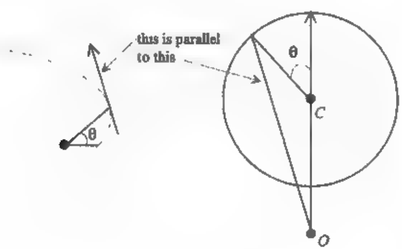
velocity diagram
(turned 90°)

In our solar system, all the planetary orbits are nearly circular. In the Earth's orbit, the distance between foci is about 1 percent of the diameter of the orbit; for Mars, it is about 9 percent; for Mercury and Pluto (whose orbits are the most eccentric), a little more than 20 percent. Halley's comet, by contrast, has an extremely eccentric elliptical orbit. The distance between its foci is 97 percent of the diameter of its orbit.

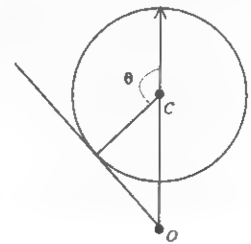
What happens if O is *outside* the circle? Let's go back to the velocity diagram before we turned it by 90°. We still have the largest velocity in the orbit at the point of closest approach:



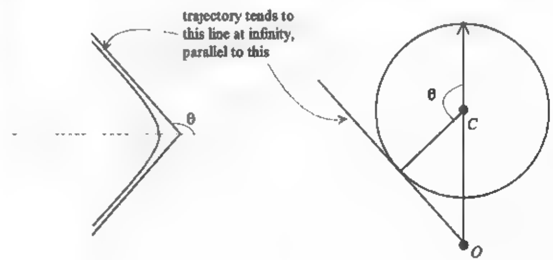
As the angle θ increases, the velocities proceed around the circle in the diagram:



At some value of θ, the line from O is the tangent to the velocity circle:



Remember, this line is also parallel to the instantaneous velocity of the orbit and the tangent to the velocity diagram is in the direction of the Δv's in the orbit diagram, which represent the changes in the velocity. In other words, at this angle θ, the change in velocity is in the same direction as the velocity itself. That means the velocity is *not* changing direction anymore. The path is no longer a curve, it is a straight line. The "orbit" is therefore not an ellipse, on which the path is never a straight line. Instead, it is a hyperbola, another of the conic sections, which tends to become a straight line far away from the focus:



On this trajectory, the "planet" falls toward the Sun from infinity, swings around, and escapes back to infinity. Its path is not an orbit at all. When it starts from infinity, and when it gets back there, its velocity is not zero, the velocity is proportional to the length of the line from O to the point where it is tangent to the velocity circle.

If the point O is *on* the circle, the "planet" also escapes to infinity, but it has zero velocity when it gets there; this trajectory is a parabola Thus, Newton's dynamics together with an inverse-square force give circular velocity diagrams. Depending on where the origin of the velocity diagram is, the orbit can be a circle, an ellipse, a parabola, or a hyperbola—the curves collectively known as the conic sections.

In the very last part of his lecture (just because he has time left over, he says), Feynman turns the machinery he's developed onto a very different kind of problem—and again, one of vast historical significance.

In 1910, two researchers, Ernest Marsden and Hans Geiger, acting at the suggestion of their leader, Ernest Rutherford, found that if a beam of α (alpha) particles (the nuclei of helium atoms) was directed at a thin gold foil, a few of them would be scattered backward instead of passing through the foil. The experiment might be thought of as crudely analogous to some alien being firing a comet into the solar system in an attempt to determine whether the mass of the solar system was spread out in a uniform blob or mostly concentrated in a compact object (the Sun) at the center. Only a compact object could have any hope of turning the comet around and hurling it back Instead of a comet, Rutherford's group had the α particle, and instead of the solar system, atoms of gold. The question was whether the matter inside an atom was spread out more or less uniformly (as current theory then held) or was concentrated at the center. The fact that some α particles were scattered backward showed that the mass had to be concentrated at the center, and this experiment constituted the discovery of the atomic nucleus.

Here, the force operating between the projectile and the constituents of the system was not gravity but electricity. Electricity is a force that acts between positive and negative electric charges (terms coined by a self-educated Newtonian scientist of the eighteenth century, Benjamin Franklin). Like gravity, electricity is an R^{-2} force that acts along the line joining the charges; unlike gravity, it can either attract charges toward each other (opposite charges) or cause charges to repel each other (like charges). The force of gravity always attracts, never repels. The electric force is vastly more powerful than the gravitational force. In fact, it is so powerful that it is self-neutralizing. Every atom in the gold foil has exactly the same amount of positive and negative charge, so from the outside the atom is neutral, exerting no electric force if it is not disturbed. The question is: What happens when an electrically charged projectile—the α particle, which is electrically positive—is fired into an atom? The answer is that it is repelled by the atomic nucleus, which contains all the positive charge and nearly all the mass of the atom. Occasionally, by sheer chance, an α particle will come close enough to the nucleus to get kicked almost directly backward. That's what Marsden and Geiger observed.

Because electricity is an R^{-2} force acting along the line between the charges, then if the particles obey Newtonian dynamics all the geometric arguments that Feynman used earlier are applicable to this problem. This problem is to find the probability that a projectile will be kicked back, so that the experiment can be compared to a quantitative theory. The starting point is the velocity-diagram circle (good for any R^{-2} force along the line between the particles), with the origin outside the circle. The "orbits" of the α particles will not be ellipses trapped forever in the vicinity of the nucleus, but rather hyperbolas, which will send the α particles away to infinity after bending their trajectories through some larger or smaller angle. We will not try to follow all the steps now, because Feynman no longer feels constrained to stick to geometrical arguments. Instead he pulls out all the analytic stops in order to arrive at what is, as he says, a very famous formula.

It deserves its fame, because it led directly to the discovery of quantum mechanics, and hence to the overthrow of the Newtonian dynamics used to arrive at the formula! But that's a story for another book. Now the time has come to put ourselves directly in the hands of the master. Enter Feynman.

4

"The Motion of Planets Around the Sun"

(MARCH 13, 1964)

The title of this lecture is "The Motion of Planets Around the Sun." . . . After the bad news you just heard announced, I have some good news for the same reason, that since the exam is coming up Tuesday, nobody wants to give a lecture that you have to study, so I'm giving a lecture that's just for the fun of it, for your entertainment [applause]. All right, all right, I won't be able to give it. Save all that for the end and then make up your mind.

The history of our subject of physics [arrived] at one of the most dramatic moments when Newton suddenly understood so much from so little. And the history of this discovery is of course the long story about Copernicus, Tycho [Brahe] making his measurements of the positions of the planets, and Kepler finding the laws which empirically describe the motion of these planets. It was then that Newton discovered that he could understand the motion of the planets by stating another law. And you know all this from the lecture on gravitation, so I continue directly from there with a quick summary of that material.

In the first place, Kepler observed that the planets went in ellipses around the Sun, with the Sun as the focus of the ellipse. He also observed—he had three observations to describe the [orbits]—that the area that's swept out by a line drawn from the Sun to the orbit is proportional, this area here, is proportional to the time. Finally, to connect planets in different orbits, he discovered that the planets with different orbits have periods, or times of rotation around the complete orbit, which bear a 3/2 power ratio to the major axis of the ellipse. If there were circles (to make it easy), it would mean that the square of the time to go around the circle is proportional to the cube of the radius of the circle.

Now, Newton was able to discover two things from this. First he noticed that equal areas and equal times meant, from his point of view about inertia, that the material would continue in a straight line at a uniform velocity if it were not disturbed, that the deviations from the uniform velocity are always directed toward the Sun, and that equal areas and equal times is equivalent to the statement that the forces are toward the Sun. So he used one of Kepler's laws already to deduce that the forces were toward the Sun. And then it is easy to argue—especially for the special case of circles from the third law—that for such circles the force which would be directed toward the Sun would have to go inversely as the square of the distance.

The reason for that is something like this. Suppose that we take a certain fractional part of an orbit, some fixed angle, a small angle, and a particle has a certain velocity in one part of the orbit and another velocity later on. Then the changes in velocity for a fixed angle are evidently proportional to the velocity. And the change in velocity during an interval of time—during a fixed time—which is the force, is evidently proportional to the velocity in the orbit times the time that it takes to go across this fraction of the orbit. I mean, divided by the time. So the velocity changes proportional to the velocity. And the time over which that change has taken place is proportional to the time that it takes to go around the whole orbit—because it is a fixed angle, like one-hundredth of the orbit. Therefore the centripetal acceleration, or change per

second of the velocity in the direction of the center, is proportional to the velocity on the orbit divided by the time that it takes to go around.[1]

You can put that in many different ways, because of course the time it takes to go around is related to the velocity by this relation. That the speed times the time is the distance around—or, rather, that the speed times the time is proportional to the radius. And so you can either substitute for the time, obtaining your famous v^2/R. Or better, I'll substitute for the velocity R/T. The velocity is evidently proportional to the radius divided by the time that it takes to go around, so that the centrifugal acceleration goes as the radius and inversely as the square of the time to go around. But Kepler tells us that the time to go around squared is proportional to the cube of the radius. That is, the denominator is proportional to the cube of the radius, and therefore the acceleration toward the center is inversely as the square of the distance. So Newton was able to deduce—in fact, [Robert] Hooke deduced earlier than Newton in the same way—that this force would be inversely as the square of the distance. So from two of Kepler's laws, we come [away] with only two conclusions. No one can verify anything that way. This may be of no particular interest, because the number of hypotheses entered is equal to the number of facts checked as the number of guesses used.

On the other hand, what Newton discovered—and which was the most dramatic of his discoveries—was that the third law [Feynman means the First Law] of Kepler was now a consequence of the other two. Given that the force is toward the Sun, and given that the force varies inversely as the square of the distance, to calculate that subtle combination of variations and velocity to determine the shape of the orbit and to discover that it is an ellipse is Newton's contribution, and therefore he felt that the science was moving forward, because he could understand three things in terms of two.

As you well know, he understood ultimately many more than three things—that the orbits in fact are not ellipses, that they perturb each other, that the motion of the Jupiter satellites is also understood, the motion of the Moon around the Earth and so on, but let us just concen-

[1] Feynman is saying $\Delta v/\Delta t$ is proportional to v/T. See Chapter 3, page 108. He refers to $\Delta v/\Delta t$ as "the centripetal acceleration" above, and below he calls it "the centrifugal acceleration."

trate on this one item, in which we disregard the interactions of one planet with another.

I can summarize what Newton said and in this way about a planet: that the changes in the velocity in equal times are directed toward the Sun, and in size they are inversely as the square of the distance. It is now our problem to demonstrate—and it is the purpose of this lecture mainly to demonstrate—that therefore the orbit is an ellipse.

It is not difficult, when one knows the calculus, and to write the differential equations and to solve them, to show that it's an ellipse. I believe in the lectures here—or at least in the book—[you] calculated the orbit by numerical methods and saw that it looked like an ellipse. That's not exactly the same thing as *proving* that it is exactly an ellipse. The Mathematics Department ordinarily is left the job of proving that it's an ellipse, so that they have something to do over there with their differential equations. [Laughter]

I prefer to give you a demonstration that it's an ellipse in a completely strange, unique, [and] different way than you are used to. I am going to give what I will call an elementary demonstration. [But] "elementary" does not mean easy to understand. "Elementary" means that very little is required to know ahead of time in order to understand it, except to have an infinite amount of intelligence. It is not necessary to have knowledge but to have intelligence, in order to understand an elementary demonstration. There may be a large number of steps that are very hard to follow, but each step does not require already knowing calculus, already knowing Fourier transforms, and so on. So by an elementary demonstration I mean one that goes back as far as one can with regard to how much has to be learned.

Of course, an elementary demonstration in this sense could be first to teach [you] calculus and then to make the demonstration. This, however, is longer than a demonstration which I wish to present. Secondly, this demonstration is interesting for another reason—it uses completely geometrical methods. Perhaps some of you were delighted in geometry in school with the fun of trying or having the ingenuity to discover the right construction lines. The elegance and beauty of geometrical demonstration is often appreciated by lots of people. On the other hand, after Descartes, all geometry can be reduced to algebra, and today all

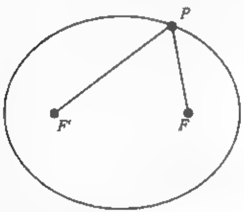
mechanics and all these things are reduced to analysis with symbols on pieces of paper and not by geometrical methods.

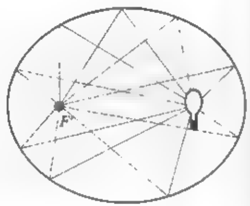
On the other hand, in the beginning of our science—that is, in the time of Newton—the geometrical method of analysis in the historical tradition of Euclid was very much the way to do things. And as a matter of fact, Newton's *Principia* is written in a practically completely geometrical way—all the calculus things being done by making geometric diagrams. We do it now by writing analytic symbols on the blackboard, but for your entertainment and interest I want you to ride in a buggy for its elegance, instead of in a fancy automobile. So we are going to derive this fact by purely geometrical arguments—well, by essentially geometrical arguments, because I don't know what that means, anything precise I don't know what it means, like purely geometrical arguments—but essentially geometrical arguments, and see how well we get on.

So our problem is to demonstrate that if this is true—that the changes in velocities are directed toward the Sun, and they are inversely as the square of the distance in equal times—that the orbit is an ellipse. We then have first to understand—we must start with something—we first must know what an ellipse is. If there is no available definition of an ellipse, it is going to be impossible to demonstrate the theory. And furthermore, if you cannot understand the meaning of this proposition, of course you also cannot demonstrate the theorem. So, many people have said, "Oh yeah, but you've got to know something about an ellipse." I know—you can't state the statement otherwise. And also you have to have some understanding of this idea. That's also true. But beyond that, I don't think we need much extra knowledge, but a large amount of attention, please, and careful thinking. That's not easy, and it's quite a job, and it's not worthwhile. It is much easier to do it by the calculus, but you're going to do it that way anyway, and you must remember that this is just to see how it would look.

There are several ways of defining an ellipse, and I have to choose one, and I will suppose that the one with which everyone is familiar is the fact that an ellipse can be made, or the ellipse is the curve that can be made, by taking one string and two tacks and putting a pencil here and going around. Or mathematically, it is the locus (nowadays they say the set of all points)—all right, the set of all points—such that the sum of the distance FP and the distance $F'P$ [F and F' being] the two fixed points, remains constant. I suppose you know that's the definition of an ellipse. You may have heard another definition of an ellipse: if you wish, these two points are called the foci, and this focus means that light emitted from F will bounce to F' from any point on the ellipse.

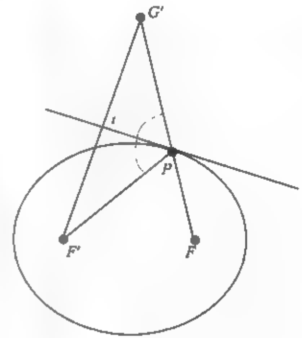
Let me just demonstrate the equivalence of those two propositions, at least. So the next step is to demonstrate that light will be reflected from F to F'. The light is reflected as though the surface here were a plane tangent to the actual curve. What I therefore have to demonstrate is this—and you know, of course, that the law of reflection for light from a plane is that the angle[s] of incidence and reflection are the same. Therefore, what I have to prove is this: that if I were to draw a line here, such that its angles made with the two lines FP and $F'P$ are equal, that that line is then tangent to the ellipse.

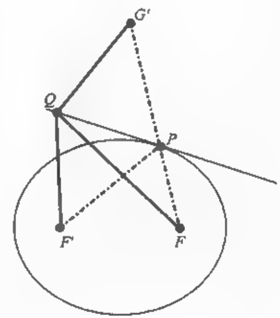




Proof: Here's the line drawn as described. Make the image point of F' in this line. That is to say, extend the perpendicular from F' to the line the same distance on the other side, to obtain G', the image of F'. Now connect the point P to G'. Notice [that] because of the equal angles, that this angle here is the vertical angle. Well, this angle is equal to this angle, because these two right triangles are exactly the same. It's an image, so this side is the same as that side, and these two angles are equal; this is a straight line. So that PG' here is exactly equal to the $F'P$ part, and incidentally, FG' is a straight line, so that the $FP + F'P$, which is the sum of these two distances, is in fact $FP + G'P$,

because $F'P = G'P$. Now, the point is that if you take any other point on the tangent—say, Q—and you took the sum of these two distances to Q, it is easy to see that the distance $F'Q$ is, again, the same as $G'Q$. So that the sum of these two distances, $F'Q$ to F, is the same as the distance from F to Q and Q to G'. In other words, the sum of the distances from the two foci on any point on the line is equal to the distance from F to G', by going up to that point and across. Evidently larger, evidently always larger than going on the straight line across. In other words, the sum of the two distances to a point Q is greater than it is for the ellipse—for any point Q except for point P. For any point on this line, then, the sum of the distances to these two points is greater than it is for a point on the ellipse.

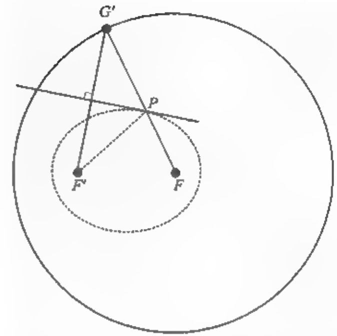




Now I take the following to be evident and perhaps you can devise a proof to satisfy you—that if the ellipse is the curve in which the sum of the two points is a constant, that the points outside the ellipse have the sum to the two points greater and the points inside the ellipse have the sum to the two points less; so that since these points on the line have a sum greater than a point on the ellipse, all this line lies outside the ellipse with the sole exception of the point P, whence it must be tangent and does not intersect at two points nor ever come inside. All right, so the thing is therefore tangent, and we know that the reflection law is right.

I have another property to describe about an ellipse, the reason for which will be completely obscure to you, but it's something which I will need later in this demonstration.

May I say that although the methods of Newton were geometrical, he was writing in a time in which the knowledge of the conic sections was the thing that everybody knew very well, and so he perpetually uses (for me) completely obscure properties of the conic sections, and I have, of course, to demonstrate my properties as I go along I would like, however, for you to take the same diagram again, which I made here, and draw it over again. It's drawn exactly the same here: F' and F, there's that tangent line, here's the image point G' of F'. However, I would like for you to imagine what happens to the image point G' as the point P goes around the ellipse. It is evident, as I already indicated, that PG' is the same as $F'P$, so that $FP + F'P$ is a constant, [and that] means that $FP + PG'$ is a constant. In other words, that FG' is a constant. In short, the image point G' runs around the point F in a circle of constant radius. All right. At the same time, I draw a line from F' to G' and I find [that] my tangent is perpendicular to it. That's the same statement as all that was before. I just want to summarize that, to remind you of a property of an ellipse, which is this: that as a point G' goes around a circle, a line drawn from an eccentric point to this point G'—this is an off-center point to the point G'—will always be perpendicular to the tangent of the ellipse. Or the other way around: the tangent is always perpendicular to the line—or a line—drawn from an eccentric point. All right, that's all, [and] we'll come back to it and we'll remember, and we will review it again, so don't worry. That's just a sum-



mary of some of the properties of an ellipse, starting from the facts. That's the ellipse.

On the other hand, we have to learn dynamics, we have to put them together. So now we have to explain what dynamics is all about. I want this proposition, that's the geometry; now the mechanics, what this proposition means. What Newton means by this is this: that if this is the Sun, for instance, the center of the attraction, and at a given instant a particle were to, say, be here, and let me suppose that it moves to another point, from A to B, in a certain interval of time. Then, [if] there were no forces acting toward the Sun, this particle would continue in the same direction and go exactly the same distance to a point c. But during this motion there's an impulse toward the Sun, which, for the purposes of analysis, we will imagine all the curves at the middle instant—in other words, at this instant. In other words, we concentrate all our impulses in an approximate way of thinking to this middle moment. And, therefore, the impulse is in the direction of the Sun, and this might represent the change in motion. That means that instead of this moving to here, it moves to a new point, which is C, which is different than c, because the ultimate motion is this motion compounded from the original plus the additional impulse given toward the center

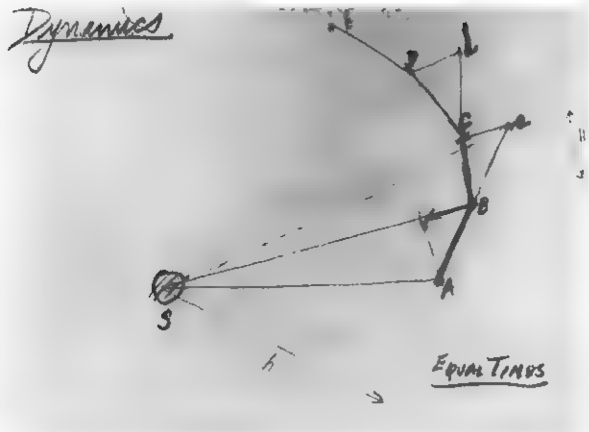


Diagram from Feynman's lecture notes.

of the Sun. So that the ultimate motion is along the line *BC*, and at the end of the second interval of moment of time the particle will be at *C*. I emphasize that *Cc* is parallel to and equal to *BV*, let us say, the impulse given from the Sun. It is therefore parallel to a line from *B* to the center of the Sun. Finally, the rest of the statement is that the size of *BV* will vary inversely as the square of the distance as we go around the orbit.

I have drawn this same thing over again here—exactly the same way, no change at all, excepting color makes it more interesting. Here's the motion that the particle would have—has in the first instant of time—and the motion which it would continue to have if it were to continue for the second interval of time with no force. May I point out to you that the areas that would be swept through in that case would be equal during those two intervals of time. For these two distances, *AB* and *Bc*, are evidently equal, and therefore the two triangles *SAB* and *SBc*, which are the two areas, will be equal: for they have equal bases and a common

altitude. If you extend the base and draw the altitude, it's the same altitude for both triangles; and since the bases are equal, the areas then swept through are equal.

On the other hand, the *actual* motion is not to the point *c* but to the point *C*, which differs from the position *c* by a displacement in the direction of the Sun at the moment *B*, that is, in the blue line parallel to the original blue line. Now I would like to point out to you that the area that would be most occupied—I mean, which would be swept out in that second interval of time even if there were a force: namely, the area *SBC*—is the same as the area that there would be if there were no force—namely, *SBc*. The reason is that we have two triangles which

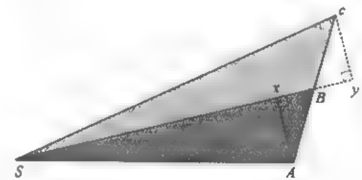
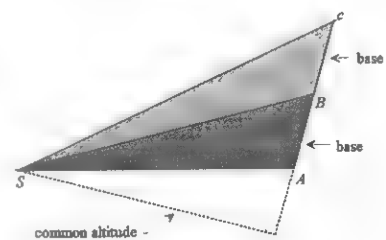
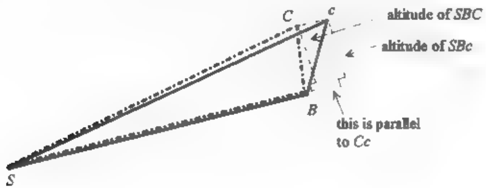


Figure in Chapter 3



Feynman does it this way instead

have a common base and who have an equal altitude, for they lie between parallel lines. Since the area[s] of the triangle *SBC* and the triangle *SAB* are equal—but since those points *A*, *B*, and *C* represented positions in succession at equal times in the orbit—we see that the area[s]moved through in equal times are equal. We can also see that the orbit remains a plane, that the point *c* being in the plane and the line *Cc* being in the plane of *ABS*, the remaining motion is in the plane *ABS*.



And I have drawn a succession of such impulses around this imaginary polygonal orbit. Of course, to find the actual orbit, we need to make the same analysis with a much smaller interval of time—and a much finer rate of impulsing—until we get the limiting case, in which we have a curve. And in the limiting case in which we have a curve—the area swept by this thing—the curve will lie in a plane, and the area swept will be proportional to the time. So that's how we know that we have equal areas in equal times. The demonstration that you have just seen is an exact copy of one in the *Principia Mathematica* by Newton, and the ingenuity and delight which you may or may not have gotten from it is that already existing in the beginning of time.

Now the remaining demonstration is not one which comes from Newton, because I found I couldn't follow it myself very well, because it involves so many properties of conic sections. So I cooked up another one.

We have equal areas and equal times. I would like now to consider what the orbit would look like if instead of using equal time, one were to think of the succession of positions which correspond to *equal angles*

from the center of the Sun. In other words, I repicture the orbit with the succession of points, *J*, *K*, *L*, *M*, *N*, which correspond not to equal instants, like they did in the diagram before, but rather [to] equal angles of inclination from the original position. To make this a little bit simpler, although it is not at all essential, I have supposed that the original motion was perpendicular to the Sun at the first point—but that's not essential, it just makes the diagrams cleaner.

Diagram from Feynman's lecture notes.

Now we know from the proposition previously that equal [areas] occupy equal times to be swept through. Now listen: I would point out to you that . . . equal angles, which is what I'm aiming for, means that areas are not equal, no, but they are proportional to the square of the distance from the Sun; for if I have a triangle of a given angle, it is clear that if I make two of them that they are similar; and the proportional area of similar triangles is proportional to the square of their dimensions.[2] Equal angles therefore means—since areas are proportional to time—equal angles therefore means that the times to be swept through these equal angles are proportional to the square of the distance. In other words, these points—*J*, *K*, *L*, and so on—do not represent

[2]This is the point explained in the footnote to Chapter 3, page 115.

pictures of the orbit at equal times, no, but they represent pictures of the orbit with successions of times which are proportional to the square of the distance.

Now, the dynamical law is that there are equal changes in velocity, no—that the changes in velocity vary inversely as the square of the distance from the Sun—that is, the changes of velocity in equal times. Another way of saying the same thing is that equal changes of velocity will occupy times proportional to the square of the distance. It's the same thing. If I take more time, I get more change in the velocity, and, although they are falling off for equal times inversely as the square, if I make my times proportional to the square of the distance, then the changes in velocity will be equal. Or, the dynamical law is. equal changes in velocity occur in times proportional to the square of the distance. But look, equal angles were times proportional to the square of the distance. And so we have the conclusion, from the law of gravitation, that equal changes of velocity will occur in equal angles in the orbit. That's the central core from which all will be deduced—that equal changes in velocity occur when the orbit is moving through equal angles. So I now draw on this diagram a little line to represent the velocities. Unlike the other diagram, those lines are not the complete line from *J* to *K*, for in that diagram those were proportional to the velocities, for the times were equal, and the length divided by equal times represented the velocities. But here I must use some other scale to represent how far the particle would have gone in a given unit of time, rather than in the times which are, in fact, proportional to the square of the distance. So these represent the velocities in succession. It is quite difficult in that diagram to find out what the changes are.

I therefore make another diagram over here, which I'll call the diagram of the velocities, in which I draw a picture on a magnified scale only for convenience. These are supposed to represent exactly these same lines. This would represent the motion per second of a particle at *J* or in a given interval of time, at *J*. This would represent the motion that a particle would've made from the beginning in a given interval of time. And, I put them all at a common origin, so that I can compare the velocities. So I have then a series of the velocities for the succession of these points.

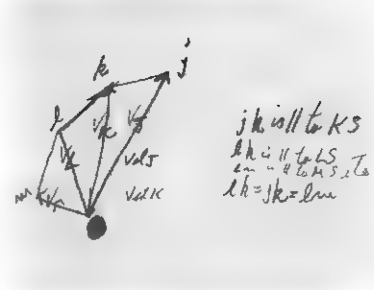
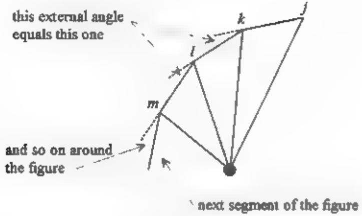


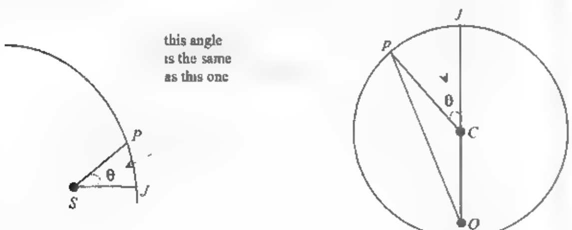
Diagram from Feynman's lecture notes.

Now, what are the changes in the velocity? The point is that in the first motion, this is the velocity. However, there is an impulse toward the Sun, and so there is a change in velocity, indicated by the green line that produces the second velocity, v_K. Likewise, there's another impulse toward the Sun again, but this time the Sun is at a different angle, which produces the next change in the velocity, v_L, and so on. Now, the proposition that the changes in the velocities were equal—for equal angles, which is the one that we deduced—means that the lengths of these succession of segments are all the same. That's what it means.

And what about their mutual angles? Since this is in the direction of the Sun at this radius, since this is at the direction of the Sun at that radius, and since this is the direction of the Sun at that radius, and so on, and since these radii each successively have a common angle to one another—so it is likewise true that these little changes in the velocity have, mutually to one another, equal angles. In short, we are constructing a regular polygon. A succession of equal steps, each turn through an equal angle, will produce a series of points on the surface underlying a circle. It will produce a circle. Therefore, the end of the velocity vector—if they call it that, the ends of these velocity points; you're not supposed to know what a vector is in this elementary description—will lie on a circle. I draw the circle again.



I review what we found out. I take the continuous limit, where the intervals of angle are very tiny indeed, to obtain a continuous curve. Let θ be the angle, total angle, to some point P, and let v_P represent the velocity of that point in the same way as before. Then the diagram of velocities will look like this. This is the origin of the velocity diagram, the same as over there, and this is the velocity vector corresponding to this point P. Then this lies on a circle, but always not necessarily the center of that circle. However, the angle that you've turned through in the circle is the same θ as here. The reason for that is that the angle turned through from the beginning by this thing is proportional to the angle turned through by the orbit, because it's the succession of the same number of small angles. And therefore, this angle in, here, is the same angle as in, here.

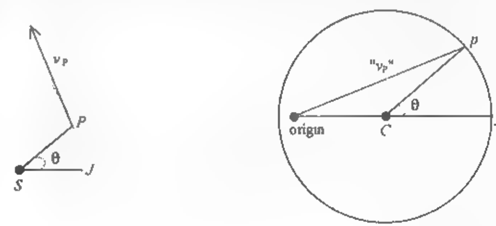


(orbit diagram)

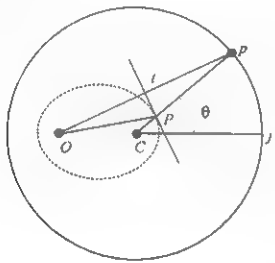
(velocity diagram)

So here is the problem, here's what we have discovered: that if we draw a circle and take an off-center point, then take an angle in the orbit—any angle you want in the orbit—and draw the corresponding angle inside this constructed circle and draw a line from the eccentric point, then this line will be the direction of the tangent. Because the velocity is evidently the direction of motion at the moment and is in the direction of the tangent to the curve So our problem is to find the curve such that if we draw a point from an eccentric center, the direction of the tangent of that curve will always be parallel to that when the angle of the curve is given by the angle in the center of that circle.

In order to make still clearer why it is going to come out in this thing, I'll turn the velocity diagram 90°, so that the angles correspond exactly and are parallel to each other. This diagram under here, then, is precisely the same diagram as the one you see above, but turned 90°—only to make it easier to think. This, then, is the velocity vector, except that it's turned 90° because the whole diagram is turned 90°. That is, this is perpendicular evidently to that, and therefore this is evidently perpendicular to that. In short, we must find the curve such that if we put the orbit in it, I think I've started—yes, so I'll just say it and then I'll draw it again—if we put the orbit in it at a given point, here, where this line intersects the orbit (never mind the scales, they're all imaginary, I mean, it's all in proportion), where this line intersects the orbit, the tangent should be perpendicular to that line from an eccentric point.



I draw it again, to show you how it is. You know now what the answer is. But here's a picture again of the same velocity circle, but this time the orbit is drawn inside at a different scale, so that we can see this picture laid right over this picture, so the angles correspond. So since the angles correspond, I can draw the single line to represent both the point P on the orbit and the point p on the velocity circle. Now what we have discovered is that the orbit is of such a character that a line drawn from the eccentric point—here, from an extension of this point onto a circle outside—will always be perpendicular to the tangent to the curve. Now that curve is an ellipse, and you can find that out by the following construction.

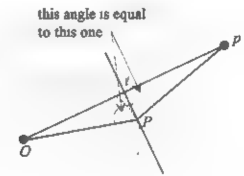


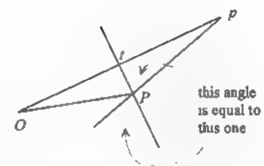
Construct the following curve. The curve I'm going to construct will satisfy all the conditions. Construct the following curve. Always take the perpendicular bisector of this line and ask for its intersection with the other line, Cp, and call that intersection point P This is the perpendicular bisector. Now I'll prove two things. First, that the locus of this point that's been generated there is an ellipse, and, second, that this line is a tangent there, too—that is, to the ellipse—and therefore satisfies the conditions, and all is well.

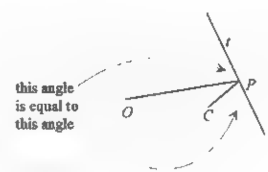
First, that it's an ellipse: Since this was the perpendicular bisector, it is at equal distances from O and p. It is therefore clear that Pp is equal to PO. That means that $CP + PO$, which is therefore equal to

$CP + Pp$, is the radius of the circle, which is evidently constant. So the curve is an ellipse, or the sum of these two distances is a constant.

And next, this line is tangent to the ellipse because, since . . . the two triangles are congruent, this angle here is equal to this angle here.







But if I extend this line on the other side, [then] also is that angle equal. So therefore the line in question makes an equal angle with the two lines to the foci. But we proved that that was one of the properties of an ellipse—the reflection property. Therefore, the solution to the problem is an ellipse—or the other way around, really, is what I proved: that the ellipse is a possible solution to the problem. And it is this solution. So the orbits are ellipses. Elementary, but difficult.

I have considerable more time, and so I will say a few things about this. In the first place, I would like to say how I got this demonstration—the fact that the velocities went in a circle. The demonstration [of] this point was due to Mr. Fano and I read it. And after that, to prove that it was an ellipse took me an awful long time: that is, the obvious, simple step—you turn it this way, and you draw that and all that. Very hard, and like all these elementary demonstrations they require a large amount—like any geometrical demonstration—of ingenuity. But once presented, it's elegantly simple. I mean, it's just finished. But the fun of it is that you've made a kind of a carefully put-together piece of pieces.

It is not easy to use the geometrical method to discover things. It is very difficult, but the elegance of the demonstrations after the discoveries are made is really very great. The power of the analytic method is that it is much easier to discover things than to prove things. But not in any degree of elegance. It's a lot of dirty paper, with x's and y's and crossed out, cancellations and so on.

I would like to point out a number of interesting cases. It of course can happen that the point O lies on the circle, or even that the point O lies outside the circle. It turns out that the point O lying on the circle does not produce, of course, an ellipse; it produces a parabola. And the point O lying outside the circle, which is another possibility, produces a different curve, a hyperbola. I leave some of those things for you to play with. On the other hand, I would like now to make some application of this and to continue the argument that Mr. Fano originally made, for another purpose. He was going in a different direction, and I'd like to show you that.

What he [Fano] was trying to do was to make an elementary demonstration of a law which was very important in the history of physics in

1914. And that had to do with the so-called Rutherford's law of scattering. If we have an infinitely heavy nucleus—which we don't have, but suppose—and if we shoot a particle by that nucleus, then it will be repelled by an inverse-square law, because of the electrical force. If q_e is the charge on an electron, then the charge on the nucleus is Z times q_e when Z is the atomic number. Then the force between the two things is given by $4\pi\epsilon_0$ times the square of the distance, which for simplicity I will write temporarily as z/R^2—the constant over R^2. I don't know whether you've done this in the class or not; but I'll suppose, I'll define another thing because, $q_e^2 / 4\pi\epsilon_0$ will be written e^2 for short. Then this thing is just Ze^2 / R^2. Anyway, that's the force inversely as the square of the distance, but it's a repulsion. And now the problem is the following: If I shoot a lot of particles at these nuclei, where I can't see the nuclei, how many of them will be deflected through various angles? What percentage will be deflected more than 30°? What percentage will be deflected more than 45°? And how are they distributed in angles? And that was the problem that Rutherford wanted solved, and when he had the correct solution, he then checked it against experiment.

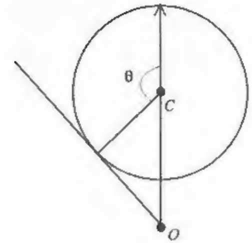
[At this point, Feynman goes off in the wrong direction. He'll correct himself in a moment.]

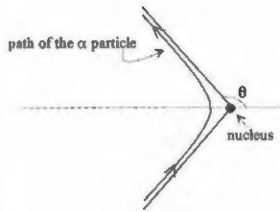
And he found that the ones that were supposed to be deflected through large angles were not there. In other words, the number of particles deflected through large angles was much less than you would think, and he therefore deduced that the force was not as strong as $1/R^2$ for small distances. Because it is obvious that to get the large angle, you need a lot of force, and it corresponds to the [particles] that hit [the nucleus] almost head-on. So those which come very close to the nucleus do not seem to come out the way they ought to, and the reason is that the nucleus has a size . . . I've got the story backwards. If the nucleus had a big size, then those which were supposed to come out at large angles wouldn't get their full force, because they would get inside the charge distribution and would be deflected less. I got mixed up. Excuse me. I start again.

Rutherford deduced how it should go if all the forces were concentrated at the center. In his day, it was supposed that the charge in an atom was distributed uniformly over the atom, and in order to discover this distribution, he thought that if he scattered these particles, they would show a weaker deflection—they would never show a very large deflection corresponding to a very close approach to the repulsion center because [in] the close approach there's no center. He, however, did find the large-angle deflections, and deduced that the nucleus was small and that the atom had all its mass at a very small central point. I got it backwards. It was later that it was demonstrated, by the same thing again, that the nucleus has a size. But the first demonstration was that the atom is not as big, for this kind of electrical purposes, as the whole atom is known to be: that is, all the charge was concentrated at the center, and thus the nucleus was discovered. However, we need now to understand this: we need to know what the law is for the angle of deflection here, and that we can obtain in this way.

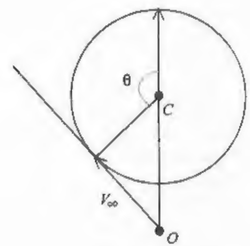
Suppose that we do the same thing as we did before, and we draw the orbit. Here is the charge, and here is the motion of a particle going around, only this time it's repulsion. I start the picture at this point, for the fun of it, and I draw my velocity circles as before. This is the velocity. We know that the velocity, the initial velocity at this point—I should use the same colors so you know what I'm doing, this should be blue, this orbit is red—now the velocity changes lie on a circle. But the changes in the velocity this time are repulsions, and the sign is reversed. And after some minor thought, you can see that the deflections go like that, and that the center of the calculation [which] used to be called the origin of the velocity space O, lies on the outside of the circle. And the succession of small velocity changes lie on the circle, and the succession of velocities then in the orbit are these lines, until a very interesting point comes: until we get to this tangent.

At this tangent point to the curve—what does it mean? It means that all the changes in velocity are in the direction of the velocity. But the changes in the velocity are in the direction of the Sun, and that means that this velocity, in this part of the diagram, is in the direction of the Sun, because it is in the direction of the changes. That is to say, this point here, as we approach this point here—which I could call x, say—corresponds to coming from infinity toward the Sun along a line here. That is, very far out we are directed toward the Sun very closely (not the Sun, but the nucleus) and then as it comes around here—this diagram





("orbit" diagram)



(velocity diagram)

should be the other way, the arrows should be here, I got the changes the wrong way in time—comes around here and goes out this way and, going out that way, corresponds to going with the velocity off in this direction.

Now, if we draw then the orbit more carefully, it will look very much like this. It goes around like this. If I call this point, here, V_∞ then the velocity that the particle has at the beginning is V_∞. If, on the same scale, I call the radius of this circle V—the velocity corresponding to the radius of the circle—I'm going to make up some equation, I'm not going to do it completely geometrically, but to save time and so on, I've done all the work. One should not ride in the buggy all the time. One has the fun of it and then gets out. Now first I want to find the velocity of the center, the radius of the velocity circle. In other words, I'm now going to come down and make some of these geometric things more analytic.

I will suppose that the force is some constant: the force—the acceleration, rather—is some constant over R^2. For gravity, this constant is GM and, for electricity, it is Ze^2/m, over m because of the acceleration. That is to say, the changes in velocity are always equal to z/R^2 times the time. Now let us suppose that we call α, which is a constant for the motion, the area swept by the orbit per second. That is then this way: that the time—if I wanted to change this to angle, I have the following—$R^2\Delta\theta$ would be the area. If I divide that by the rate that area is swept through—this tells me how much time it takes to sweep an angle. The time is, then, for given angles, proportional to the square of the distance. All this I'm saying now analytically, where I said in words before. Substitute this Δt, in here, to find out how the changes in the velocity are with respect to angle, and one obtains $R^2\Delta\theta/\alpha$, or the R^2's cancel, and it means that the changes in velocity are as advertised: for equal angles, equal.

Now then, the velocity diagram—although this isn't the piece of the orbit that you can get to, never mind—these are changes in the velocity and these are changes in the angle in the orbit. So ΔV is also equal by the geometry of that circle to the radius of the circle, which I call $V_R \times \Delta\theta$. In other words, we have that the radius of the velocity circle is equal to z/α, where α is the rate of area swept per second and z is a constant having to do with the law of force. Now, the angle through which this planet has deflected is this one, here, and I call it, the angle of deflection from the planet—I mean the charged particle from the nucleus. It is evident from my discussion that it's the same as this angle in here, ϕ, because these velocities are parallel to the two original directions. It is clear, therefore, that we can find ϕ if we can get the relation with V_∞ and V_R. You see, look, tangent of $\phi/2 = V_R/V_\infty$ and that gives us the angle. The only thing is that we need—we have to substitute for V_R, $z/\alpha R$, and we have that much.

Now, it doesn't do us much good until we know α for this orbit. An interesting idea is this: think of this thing as approaching this, so that if there were no force it would miss by a certain distance, b. This is called the impact parameter. We imagine that the thing comes from infinity aimed for the force center, but is missing—because it misses, it is deflected. By how much is it deflected, if it was aimed to miss by b? That's the question. If it's aimed to miss by a distance b, how much will it get deflected?

So I need now only determine how α is related to b. V_∞ is the distance gone in 1 second, so if I were to draw way out here a horrible-looking area, a triangle—a terrible-looking triangle, then the—I got a factor of 2 somewhere, yeah, the area of a triangle is 1/2 R^2. There are two factors, two, which you will straighten out please when the time comes. There is 1/2 in here and, there is 1/2 somewhere else, which I'm now going to make. The area of this triangle is the base V_∞ times the height b times 1/2. Now that triangle is a triangle through which a particle would sweep—the radius would sweep in 1 second. And this is, therefore, α. So, therefore, we have that this goes as z/bV_∞^2. That tells us that given the impact distance, the aiming accuracy, what angle we would find in the deflection in terms of the speed at which the particle approaches and the known law of force. So it's completely finished.

One more thing that is rather interesting. Suppose that you would like to know with what probability, what chance is there of getting a deflection more than a certain amount. Let's say you pick a certain ϕ—ϕ_0, say—and you want to make sure that you get greater than ϕ_0. That only means that you have to hit inside an area closer than the b which belongs to that ϕ. Any collision closer than b will produce a deflection bigger than ϕ_0, where b is b_0, belonging to ϕ_0 through this equation. If you come further away, I have less deflection, less force. So, therefore, the so-called cross section of area that you have to hit for deflection, to be greater than ϕ (I'll leave off the naught), is πb^2, where b is $z/V_\infty^2 \tan^2 \phi/2$. In other words, it is $\pi z^2/V_\infty^4 \tan^2 \phi/2$. And that's the law of Rutherford's scattering. That tells you the probability of the area you have to hit—the effective area that you have to hit—in order to get a deflection more than a certain amount. This z is equal to Ze^2/m; this is a fourth power, and it is a very famous formula.

It is so famous that, as usual, it was not written in this form when it was first deduced, and so I, just for the famousness of it, will write it in a form—well, I'll leave you to write it in a form. I'll write just the answer, and I'll let you see if you can show it. Instead of asking for the cross section for a deflection greater than a certain angle, we can ask for the piece of cross section, $d\sigma$, that corresponds to the deflection in the range $d\phi$ that the angle should be between, here, and there. You just have to differentiate this thing, and the final result for that thing is given as the famous formula of Rutherford, which is $4\,Z^2e^4$ times $2\pi \sin\phi \, d\phi$ divided by $4m^2\,V_\infty^{\,4}$ times the sine of the fourth power of $\phi/2$. This I write only because it's a famous one that comes up very much in physics. The combination $2\pi \sin\phi \, d\phi$ is really the solid angle that you have in range $d\phi$. So in a unit of solid angle, the cross section goes inversely as the fourth power of the sine of $\phi/2$. And it was this law which was discovered to be true for scattering of α particles from atoms, which showed that the atoms had a hard center in the middle . . . a nucleus. And it was by this formula that the nucleus was discovered.

Thank you very much.

Epilogue

Richard Feynman conjured up his own brilliant proof of the law of ellipses, but he was not the first to think of it. The same proof, right down to the crucial insight of turning the velocity diagram on its side, appears in a little book called *Matter and Motion*, written by James Clerk Maxwell and first published in 1877. Maxwell attributes the method of proof to Sir William Hamilton, a name familiar to all physicists. (The Hamiltonian is a crucial element of quantum mechanics.) Apparently, Hamilton was the first to use the velocity diagram, which he called the Hodograph, to study the motion of a body. In his lecture, Feynman generously credits a mysterious "Mr. Fano" with the idea of the circular velocity diagram. He is referring to a book by U. Fano and L. Fano, *Basic Physics of Atoms and Molecules* (1959), where a circular velocity diagram is used to derive the Rutherford scattering law presented by Feynman at the end of his lecture. If Fano and Fano knew about Hamilton and his Hodograph, they do not say so.

Hamilton was part of a centuries-long tradition of refining Newton's mechanics into formulations of ever greater sophistication and elegance. For more than two hundred years after the publication of the *Principia*, the universe of Newton reigned supreme. Then, early in the twentieth